A NEW CLASS OF AFFINE $K(\pi, 1)$ ARRANGEMENTS

KATHERINE GOLDMAN† AND JINGYIN HUANG*

ABSTRACT. We show that a certain class of affine hyperplane arrangements are $K(\pi, 1)$ by endowing their Falk complexes with an injective metric. This gives new examples of infinite $K(\pi, 1)$ arrangements in dimension $n > 2$.

1. INTRODUCTION

Let $\mathcal{A}$ be an *affine hyperplane arrangement* in $\mathbb{R}^n$, i.e., a locally finite collection of affine hyperplanes in $\mathbb{R}^n$. We consider the complex manifold which is the complement of the following collection of hyperplanes in $\mathbb{C}^n$:

$$M(\mathcal{A}) = \mathbb{C}^n - \bigcup_{H \in \mathcal{A}} (H \otimes \mathbb{C}).$$

It is an important question to understand the topology of $M(\mathcal{A})$, see e.g. [FR86, FR98]. We will be specifically interested in the asphericity of $M(\mathcal{A})$. If the manifold $M(\mathcal{A})$ is aspherical, we call $\mathcal{A}$ a $K(\pi, 1)$ *arrangement*.

Unlike the situation of knot complements in $\mathbb{S}^3$, asphericity of $M(\mathcal{A})$ is a relatively rare phenomenon. However, there are some specific classes of $\mathcal{A}$ where asphericity is known, for example:

(1) $\mathcal{A}$ is central and simplicial by Deligne [Del72];
(2) $\mathcal{A}$ is supersolvable by Terao [Ter86];
(3) $\mathcal{A}$ is certain type of line arrangement in $\mathbb{R}^2$ by Falk [Fal95];
(4) $\mathcal{A}$ is the collection of reflection hyperplanes associated with an affine Coxeter group by Paolini and Salvetti [PS21].

These results are obtained through different means: (1) and (4) rely heavily on Garside theory; (2) is obtained through a fibration argument; (3) uses a form of conformal non-positive curvature for 2-dimensional complexes, allowing one to compute the second homotopy group directly. Given that there are relatively few methods and examples of aspherical arrangements when $n \geq 3$, it is desirable to extend Falk's method over dimension 2, which is the goal of this article. In higher dimensions, we must use a different notion of non-positive curvature in place of the conformal non-positive curvature in [Fal95] which can only be used in dimension 2.

Given an affine arrangement $\mathcal{A}$, an $\mathcal{A}$-*vertex* is a point in $\mathbb{R}^n$ which can be realized as intersection of elements of $\mathcal{A}$. The *local arrangement* at an $\mathcal{A}$-vertex x is the collection of all hyperplanes in $\mathcal{A}$ that contain x. An interesting feature of Falk's result, is the local-to-global phenonmenon that for certain classes of arrangements $\mathcal{A}$, one can detect the asphericity of $M(\mathcal{A})$ by looking at the combinatorial features of its local arrangements. Motivated by this, we consider the following class of arrangements characterized by their local arrangements.

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Definition 1.1. We say an affine hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is *admissible*, if at each $\mathcal{A}$-vertex x, the local arrangement at x is a translate of the following four types:

(1) (type B_n) $x_i \pm x_j = 0$ for $1 \leq i \neq j \leq n$ and $x_i = 0$ for $1 \leq i \leq n$;
(2) (type D_n) $x_i \pm x_j = 0$ for $1 \leq i \neq j \leq n$;
(3) (skewed type A_n) $x_i = 0$ for $1 \leq i \leq n$ and $x_i = x_j$ for $1 \leq i \neq j \leq n$, or any image of this this arrangement under the $(\mathbb{Z}/2\mathbb{Z})^n$ action on $\mathbb{R}^n$ by reflections about the coordinate hyperplanes;
(4) or a product of the previous types.

Note that any affine Coxeter arrangement associated with a non-exceptional affine Coxeter group (i.e., types $\widetilde{A}_n, \widetilde{B}_n, \widetilde{C}_n, \widetilde{D}_n$) is an admissible arrangement[1]. Although most of the arrangements in Definition 1.1 are not Coxeter arrangements.

Theorem 1.2. *Let $\mathcal{A}$ be an admissible affine arrangement in $\mathbb{R}^n$ which is invariant under the action of a discrete translation subgroup $\mathbb{Z}^n$ of $\mathbb{R}^n$ (this does not have to be the usual embedding of $\mathbb{Z}^n$). Suppose $n \leq 4$. Then $\mathcal{A}$ is a $K(\pi, 1)$ arrangement. More generally, modulo a group theoretical conjecture on the spherical Artin group of type D_n (Conjecture 2.16), $\mathcal{A}$ is a $K(\pi, 1)$ arrangement for any n.*

In the situation of the above theorem, we have a free action of $\mathbb{Z}^n$ on $M(\mathcal{A})$. Then the fundamental group of $M(\mathcal{A})/\mathbb{Z}^n$ can be viewed as a generalization of the affine Artin groups (when $\mathcal{A}$ is an affine Coxeter arrangement, this gives a finite index subgroup of the corresponding affine Artin group).

Corollary 1.3. *Under the assumption of Theorem 1.2, the manifold $M(\mathcal{A})/\mathbb{Z}^n$ is homotopy equivalent to a finite aspherical cell complex, which is the quotient of the Salvetti complex of $\mathcal{A}$ (*[Sal87]*) by a free action of $\mathbb{Z}^n$. In particular, the fundamental group of $M(\mathcal{A})/\mathbb{Z}^n$ is of type F.*

Theorem 1.2 is a special case of a more general statement which does not require $\mathcal{A}$ to be $\mathbb{Z}^n$-invariant, see Theorem 3.17.

Now we give more concrete examples of arrangements where Theorem 1.2 (or more specifically, Theorem 3.17) applies. For each dimension n, we will construct an infinite family of finite affine arrangements $\mathcal{H}_{k,n}$, and an infinite family of $\mathbb{Z}^n$-invariant affine arrangements $\mathcal{K}_{k,n}$. These arrangements are not Coxeter arrangements, and their complexified complement do not admit iterated fibration structure in an obvious way. The finite arrangements are not simplicial. So the $K(\pi, 1)$ results for these arrangements are new.

Definition 1.4. For $k \geq 1$, let $\mathcal{H}_{k,n}$ be the affine hyperplane arrangement in $\mathbb{R}^n$ given by $x_i \in \{-2k-1, -2k+1, \ldots, -3, -1, 1, 3, \ldots, 2k-1, 2k+1\}$ for $1 \leq i \leq n$ and $x_i \pm x_j = 0$ for $1 \leq i \neq j \leq n$. This family $\mathcal{H}_{k,n}$ generalizes Falk's [Fal95, Example 3.13] which is neither supersolvable or simpicial.

For $k \geq 1$, let $\mathcal{K}_{k,n}$ be the affine hyperplane arrangement in $\mathbb{R}^n$ given by $x_i \in \mathbb{Z}$ for $1 \leq i \leq n$, and $x_i + x_j \in 2k\mathbb{Z} + 1$, $x_i - x_j \in 2k\mathbb{Z}$ for $1 \leq i \neq j \leq n$.

Theorem 1.5. *(=Theorem 4.5) For $n \leq 4$ and any $k \geq 1$, the arrangements $\mathcal{H}_{k,n}$ and $\mathcal{K}_{k,n}$ are $K(\pi, 1)$ arrangements. More generally, modulo a group theoretical*

[1] We use a different description of the $\widetilde{A}_n$ arrangement, where the hyperplanes are $x_i \in \mathbb{Z}$ for $1 \leq i \leq n$ and $x_i - x_j \in \mathbb{Z}$ for $1 \leq i \neq j \leq n$. This does not affect the topology of $M(\mathcal{A})$.

conjecture on the spherical Artin group of type D_n (Conjecture 2.16), $\mathcal{H}_{k,n}$ and $\mathcal{K}_{k,n}$ are $K(\pi,1)$ arrangements for any n,k.

For $n=2$, Falk [Fal95] constructed a (locally infinite) 2-dimensional complex $\mathbb{F}_{\mathcal{A}}$ that is homotopy equivalent to the universal cover of $M(\mathcal{A})$. This construction of Falk can be generalized to higher dimensions without too much difficulty [Hua24a]. Later, we show $\mathbb{F}_{\mathcal{A}}$ is homotopy equivalent to the universal cover of $M(\mathcal{A})$ whenever all the local arrangements are $K(\pi,1)$. Thus all the above results rely on showing the complex $\mathbb{F}_{\mathcal{A}}$ is contractible.

Recall that a geodesic metric space X is *injective* if any pairwise intersecting closed metric balls in X have non-empty common intersection. For example, $\mathbb{R}^n$ equipped with the ℓ^∞ metric is injective. Injective metric spaces are contractible, and they are connected to the above theorems in the following way.

Theorem 1.6. (= Theorem 3.17 and Theorem 4.5) *Under the assumptions of any of the previous theorems, the Falk complex $\mathbb{F}_{\mathcal{A}}$ admits a metric which makes it an injective metric space.*

The reason for us to consider skewed type A_n arrangements as local arrangements, rather than the standard A_n-arrangements, is for the compatibility of arranging injective metrics on $\mathbb{F}_{\mathcal{A}}$. A substantial part of the article (Section 5) is devoted to checking such compatibility (Proposition 3.15 and Remark 3.16).

It is a topic of independent interest to produce natural examples of injective metric spaces arising from group theory, and the above theorem gives many such examples. As the above theorem becomes conditional for $n\geq 5$, we also have the following variation that is unconditional for all dimensions, where we allow the local arrangements to be a mixture of A_n-type and B_n-type. The following corollary is more interesting in terms of providing new examples of injective metric spaces, rather than $K(\pi,1)$ results, as one can prove the arrangements in the following corollary are $K(\pi,1)$ via an iterated fibration argument.

Corollary 1.7. (= Corollary 3.19) *Suppose $\mathcal{A}$ is a complete, finite shape, affine arrangement in $\mathbb{R}^n$ such that for each $\mathcal{A}$-vertex x, the local arrangement at x is a translate of one of the following three types:*

(1) (type B_n) $x_i\pm x_j=0$ for $1\leq i\neq j\leq n$ and $x_i=0$ for $1\leq i\leq n$;
(2) (skewed type A_n) $x_i=0$ for $1\leq i\leq n$ and $x_i=x_j$ for $1\leq i\neq j\leq n$, or any image of this this arrangement under the $(\mathbb{Z}/2\mathbb{Z})^n$ action on $\mathbb{R}^n$ by reflections about the coordinate hyperplanes;
(3) or a product of the previous types.

Then $(\mathbb{F}_{\mathcal{A}},d_\infty)$ is an injective metric space and $\mathcal{A}$ is a $K(\pi,1)$ arrangement.

Structure of the article. In Section 2 we collect some background material. In Section 3 we define Falk complexes of affine hyperplane arrangements and prove some of them admit injective metrics, modulo a key proposition (Proposition 3.15) about skewed A_n arrangements. Section 5 is devoted to this key proposition. Section 4 contains some new, concrete examples of affine arrangements where our results apply.

2. Preliminaries

2.1. **Hyperplane arrangements and their dual polyhedra.** An *affine hyperplane arrangement* in the vector space $\mathbb{R}^n$ is a locally finite family $\mathcal{A}$ of affine hyperplanes. Let $\mathcal{Q}(\mathcal{A})$ be the set of nonempty affine subspaces that are intersections of subfamilies of $\mathcal{A}$ (here $\mathbb{R}^n \in \mathcal{Q}(\mathcal{A})$ as the intersection of an empty family). Each point $x \in \mathbb{R}^n$ belongs to a unique element of $\mathcal{Q}(\mathcal{A})$ that is minimal with respect to inclusion, called the *support* of x. A *fan* of $\mathcal{A}$ is a maximal connected subset of $\mathbb{R}^n$ consisting of points with the same support. Denote the collection of all fans of $\mathcal{A}$ by $\mathrm{Fan}(\mathcal{A})$. Note that $\mathbb{R}^n$ is the (disjoint) union of $\mathrm{Fan}(\mathcal{A})$. We define a partial order on $\mathrm{Fan}(\mathcal{A})$ so that $U_1 < U_2$ if U_1 is contained in the closure of U_2. Let $b\Sigma_{\mathcal{A}}$ be the simplicial complex that is the geometric realisation of this poset. For each $U \in \mathrm{Fan}(\mathcal{A})$, we choose a point $x_U \in U$. This gives a piecewise linear embedding $b\Sigma_{\mathcal{A}} \subseteq \mathbb{R}^n$ sending the vertex of $b\Sigma_{\mathcal{A}}$ corresponding to U to x_U.

By [Sal87, pp. 606-607], the simplicial complex $b\Sigma_{\mathcal{A}}$ is the barycentric subdivision of a combinatorial complex $\Sigma_{\mathcal{A}}$ whose vertices correspond to the top-dimensional fans. Namely, for each vertex of $b\Sigma_{\mathcal{A}}$ corresponding to $U \in \mathrm{Fan}(\mathcal{A})$, the union of all the simplices of $b\Sigma_{\mathcal{A}}$ corresponding to chains with smallest element U is homeomorphic to a closed disc [Sal87, Lem 6], which becomes the face of $\Sigma_{\mathcal{A}}$ corresponding to U. We will sometimes view $b\Sigma_{\mathcal{A}}$ and $\Sigma_{\mathcal{A}}$ as subspaces of $\mathbb{R}^n$. For $B \in \mathcal{Q}(\mathcal{A})$, a face F of $\Sigma_{\mathcal{A}}$ is *dual* to B, if B contains the fan U corresponding to F and $\dim(B) = \dim(U)$. We equip the 1-skeleton of $\Sigma_{\mathcal{A}}$ with the path metric d such that each edge has length 1. Given vertices $x, y \in \Sigma^0_{\mathcal{A}}$, it turns out that $d(x, y)$ is the number of hyperplanes separating x and y [Del72, Lem 1.3].

Lemma 2.1 ([Sal87, Lem 3])**.** *Let $x \in \Sigma^0_{\mathcal{A}}$ and let F be a face of $\Sigma_{\mathcal{A}}$. Then there exists unique $\Pi_F(x) \in F^0$ such that $d(x, \Pi_F(x)) \leq d(x, y)$ for any $y \in F^0$.*

The vertex $\Pi_F(x)$ is called the *projection* of x to F. A hyperplane $H \in \mathcal{A}$ *crosses* a face F of $\Sigma_{\mathcal{A}}$ if H is dual to an edge of F. For an edge xy of $\Sigma_{\mathcal{A}}$, if the hyperplane dual to xy crosses F, then $\Pi_F(x)\Pi_F(y)$ is an edge dual to the same hyperplane, otherwise we have $\Pi_F(x) = \Pi_F(y)$. Thus Π_F extends naturally to a map $\Sigma^1_{\mathcal{A}} \to F^1$.

Lemma 2.2. *(*[HP25, Lem 3.2]*) Let E and F be faces of $\Sigma_{\mathcal{A}}$. Then $\Pi_F(E^0) = F'^0$ for some face $F' \subseteq F$.*

In the situation of Lemma 2.2, we write $F' = \Pi_F(E)$.

The assignment $E \to \Pi_F(E)$ gives rise to a piecewise linear map $\Pi_F \colon \Sigma_{\mathcal{A}} \cong b\Sigma_{\mathcal{A}} \to bF \cong F$.

2.2. **The Salvetti complex.** Let $V = \Sigma^0_{\mathcal{A}}$. Consider the set of pairs (F, v), where F is a face of $\Sigma_{\mathcal{A}}$ and $v \in V$. We define an equivalence relation $\sim$ on this set by $(F, v) \sim (F', v')$ whenever $F = F'$ and $\Pi_F(v') = \Pi_F(v)$. Note that each equivalence class $[F, v']$ contains a unique representative of form (F, v) with $v \in F^0$. The *Salvetti complex* $\widehat{\Sigma}_{\mathcal{A}}$ is obtained from $\Sigma_{\mathcal{A}} \times V$ (a disjoint union of copies of $\Sigma_{\mathcal{A}}$) by identifying faces $F \times v$ and $F \times v'$ whenever $[F, v] = [F, v']$ [Sal87, p. 608]. For example, for each edge $F = v_0v_1$ of $\Sigma_{\mathcal{A}}$, we obtain two edges $F \times v_0$ and $F \times v_1$ of $\widehat{\Sigma}_{\mathcal{A}}$, glued along their endpoints $v_0 \times v_0$ and $v_1 \times v_1$. We orient the edge $F \times v_0$ from $v_0 \times v_0$ to $v_1 \times v_0 = v_1 \times v_1$. Then $\widehat{\Sigma}^0_{\mathcal{A}} = V$, while $\widehat{\Sigma}^1_{\mathcal{A}}$ is obtained from $\Sigma^1_{\mathcal{A}}$ by doubling each edge. Thus each edge of the form $F \times v$ is oriented so that its endpoint is farther from v in F^1 than its starting point.

There is a natural map $p \colon \widehat{\Sigma}_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ forgetting the second coordinate. For each subcomplex Y of $\Sigma_{\mathcal{A}}$, we write $\widehat{Y} = p^{-1}(Y)$. If F is a face of $\Sigma_{\mathcal{A}}$, then $\widehat{F}$ is a *standard subcomplex* of $\widehat{\Sigma}_{\mathcal{A}}$.

Lemma 2.3. *Let E and F be faces of $\Sigma_{\mathcal{A}}$. If $[E, v_1] = [E, v_2]$, then $[\Pi_F(E), v_1] = [\Pi_F(E), v_2]$.*

Definition 2.4. Let F be a face of $\Sigma_{\mathcal{A}}$. Consider the disjoint union of V copies of the map Π_F, where $\Pi_F \times v \colon \Sigma_{\mathcal{A}} \times v \to F \times v$. It follows from Lemma 2.3 that this map factors to a map $\Pi_{\widehat{F}} \colon \widehat{\Sigma}_{\mathcal{A}} \to \widehat{F}$, which is a retraction (see [GP12, Thm 2.2]).

The following key property of $\Pi_{\widehat{F}}$ follows directly from Definition 2.4.

Lemma 2.5. *Let E and F be faces of $\Sigma_{\mathcal{A}}$. Then $\Pi_{\widehat{F}}(\widehat{E}) = \widehat{\Pi_F(E)}$.*

Let $\mathcal{A} \otimes \mathbb{C}$ be the complexification of $\mathcal{A}$, which is a collection of affine complex hyperplanes in $\mathbb{C}^n$. Define

$$M(\mathcal{A} \otimes \mathbb{C}) = \mathbb{C}^n - \bigcup_{H \in \mathcal{A}} (H \otimes \mathbb{C}).$$

It follows from [Sal87, Thm 1] that $\widehat{\Sigma}_{\mathcal{A}}$ is homotopy equivalent to $M(\mathcal{A} \otimes \mathbb{C})$, and so they have isomorphic fundamental groups. We say $\mathcal{A}$ is a $K(\pi, 1)$ *arrangement* if $M(\mathcal{A} \otimes \mathbb{C})$ is aspherical, or equivalently, the Salvetti complex $\widehat{\Sigma}_{\mathcal{A}}$ is aspherical.

2.3. **Injective metric spaces.** A *geodesic (segment)* in a metric space X is the image of an isometric embedding from a closed interval in $\mathbb{R}$ (equipped with the usual metric on $\mathbb{R}$) to X. A metric space is *geodesic*, if every pair of points are joined by a geodesic in the space. A *ball* $B(x, r)$ in a metric space is the collection of points of distance $\leq r$ from a point x, which is the center of the ball. A geodesic metric space is *injective* if every collection of balls in the metric space which have non-empty pairwise intersection also have non-empty common intersection. As an example, the ℓ^∞-norm on $\mathbb{R}^n$ induces an injective metric on $\mathbb{R}^n$. It is known that any injective metric space is contractible, as it is possible to select a geodesic joining each pair of points such that this geodesic varies continuously depending on the endpoints (see e.g. [Lan13, Prop 3.8]). Note the each ball in an injective metric space X, endowed with the induced metric from X, is itself an injective metric space (the ball is geodesic by [Lan13, Prop 3.8 (1)]).

We recall the following local-to-global criterion for injective metric space.

Theorem 2.6. [Hae21, Thm 1.14] *Let X be a geodesic metric space that is complete, simply connected and uniformly locally injective. Then X is injective.*

A metric space is *uniformly locally injective* if there is an $\varepsilon > 0$ such that each ball of radius ε is injective. The assumption of X being geodesic is not explicitly mentioned in the statement of the theorem in [Hae21]; however, it is needed in the proof. This theorem is a consequence of a deep combinatorial local-to-global theorem in [CCHO14, Thm 3.5]. An earlier version of this theorem with the extra assumption of local compactness of the metric space appears in [Mie18].

Now we recall a particular combinatorial criterion for justifying that a metric space is injective. Let P be a poset (i.e., a partially ordered set). Let $S \subseteq P$. An *upper bound* (resp. *lower bound*) for S is an element $x \in P$ such that $s \leq x$ (resp. $s \geq x$) for every $s \in S$. The *join* of S is an upper bound x of S such that $x \leq y$ for any other upper bound y of S. The *meet* of S is a lower bound of x of S such that $x \geq y$ for any other lower bound y of S. We will write $x \vee y$ for the join of two elements x and y, and $x \wedge y$ for the meet of two elements (if the join or the meet exists). A poset is *bounded* if it has a maximal element and a minimal element. P is a *lattice* if P is a poset and any two elements in P have a join and have a meet.

A chain in P is a totally (or "linearly") ordered subset, and a maximal chain is one that is not a proper subset of any other chain. A poset has rank n if it is bounded, every chain is a subset of a maximal chain, and all maximal chains have length n. For $a, b \in P$ with $a \leq b$, the *interval* between a and b, denoted by $[a, b]$, is the collection of all elements x of P such that $a \leq x$ and $x \leq b$. The poset P is *graded* if every interval in P has a rank. The *geometric realization of* P, denoted by $|P|$, is a simplicial complex whose vertex set is P, where a collection of vertices span a simplex if and only if they form a chain in P.

Definition 2.7. Let P be a poset. We say that P is *bowtie free* if for any subset $\{x_1, x_2, y_1, y_2\} \subseteq P$ made of mutually distinct elements with $x_i < y_j$ for $i, j \in \{1, 2\}$, there exists $z \in P$ such that $x_i \leq z \leq y_j$ for any $i, j \in \{1, 2\}$.

Lemma 2.8. [BM10, Proposition 1.5] *If P is a bowtie free graded poset, then any pair of elements in P with a lower bound have a join, and any pair of elements in P with a upper bound have a meet.*

Let P be a bounded graded poset. Then P is lattice if and only if it is bowtie free.

Definition 2.9. A poset P is *upward flag* if any three pairwise upper bounded elements have an upper bound. A poset is *downward flag* if any three pairwise lower bounded elements have a lower bound. A poset is *flag* if it is both upward flag and downward flag.

An n-dimensional *unit orthoscheme* of $\mathbb{R}^n$ is the convex hull of

$$v_0 = (0, 0, \dots, 0), v_1 = (1, 0, \dots, 0), v_2 = (1, 1, \dots, 0), \cdots, v_n = (1, 1, \dots, 1).$$

We endow the unit orthoscheme with the ℓ^∞-metric. Following [BM10, Sec 5] and [Hae21, Sec 1], the *ℓ^∞-orthoscheme complex* of a poset P, denoted by $|P|_\infty$, is $|P|$ endowed with the metric such that each is assigns every top dimensional simplex in $|P|$ (i.e. those corresponding to maximal chains $x_0 < x_1 < \cdots < x_n$) the ℓ^∞-metric of a unit orthoscheme with x_i corresponding to v_i.

Via a standard procedure ([BH99, p. 65]), the ℓ^∞-metric on the simplices of $|P|$ induces a *pseudometric* d on $|P|_\infty$, which we describe here for the convenience of the reader (a pseudometric satisfies all properties of a distance function of a metric space, except the distance of two points are allowed to be zero). Given $x, y \in |P|$, an *m-string* from x to y is a sequence $(x_0 = x, x_1, x_2, \dots, x_m = y)$ such that for each i, x_i and x_{i+1} are contained in a common simplex σ_i. The *length* of this string is defined to be $\sum_{i=0}^{m-1} d_{\sigma_i}(x_i, x_{i+1})$, where d_{σ_i} denotes the metric on σ_i. Then $d(x, y)$ is defined to be infimum of the lengths of all strings from x to y. In the case P is a graded poset with finite rank, there are only finitely many isometry types of simplices appearing in $|P|_\infty$, hence this pseudometric is a metric, and it is geodesic – this follows from the arguments in [BH99, p. 101, Corollary I.7.10].

Theorem 2.10. *(*[Hae21, Thm 6.3]*) Let P be a graded poset with a minimal element and finite rank such that P is bowtie free and flag. Then $(|P|_\infty, d)$ is an injective metric space.*

2.4. **Coxeter complexes and Artin complexes.** A *Coxeter diagram* Λ is a finite simplicial graph with vertex set $S = \{s_i\}_{i \in I}$ and labels $m_{ij} \in \{3, 4, \dots, \infty\}$ for each edge $s_i s_j$. If $s_i s_j$ is not an edge, we define $m_{ij} = 2$. The Artin group A_Λ is the group with generating set S and relations $s_i s_j s_i \cdots = s_j s_i s_j \cdots$, with both sides alternating words of length m_{ij}, whenever $m_{ij} < \infty$. The Coxeter group W_Λ is obtained from A_Λ by adding the relations $s_i^2 = 1$ for each i. The kernel PA_Λ of the natural homomorphism $A_\Lambda \to W_\Lambda$ is called the *pure Artin group.*

Definition 2.11. Let A_Λ be an Artin group with Coxeter graph Λ and generating set S. Its *Artin complex* Δ_Λ [CD95, GP12, CMV20] is a simplicial complex defined as follows. For each $s \in S$, let $A_{\hat{s}}$ be the standard parabolic subgroup generated by $\hat{s} = S \setminus \{s\}$. The vertices of Δ_Λ correspond to the left cosets of $\{A_{\hat{s}}\}_{s\in S}$. A collection of vertices span a simplex if and only if the corresponding cosets have non-empty common intersection. If v is a vertex of Δ_Λ which is a left coset of $A_{\hat{s}}$, we say v is *type* $\hat{s}$.

The *Coxeter complex* $\mathfrak{C}_\Lambda$ for the Coxeter group W_Λ is defined analogously, where we replace $A_{\hat{s}}$ by $W_{\hat{s}} < W_\Lambda$ generated by $\hat{s}$. If v is a vertex of $\mathfrak{C}_\Lambda$ which is a left coset of $W_{\hat{s}}$, we say v is *type* $\hat{s}$.

Lemma 2.12. *(*[CMV20, Lem 6]*) Let $\delta \subseteq \Delta_\Lambda$ be a simplex whose types of vertices are $\{\hat{s}_i\}_{i=1}^k$. Then the link* $\mathrm{lk}(\delta, \Delta_\Lambda)$ *of δ in Δ_Λ is isomorphic $\Delta_{\Lambda'}$ where Λ' is the induced subdiagram of Λ spanned by vertices in $\Lambda \setminus \{s_1, s_2, \dots, s_k\}$.*

Now consider the special case W_Λ is a finite Coxeter group with its canonical representation $\rho\colon W_\Lambda \to \mathbf{GL}(n, \mathbb{R})$ [Dav08, Chap 6.12]. A *reflection* of W_Λ is a conjugate of $s \in S$. Each reflection pointwise fixes a hyperplane in $\mathbb{R}^n$, which we call a *reflection hyperplane.* Let $\mathcal{A}$ be the family of all reflection hyperplanes. The hyperplane arrangement $\mathcal{A}$ is the *reflection arrangement* associated with W_Λ. We denote $\Sigma_\Lambda = \Sigma_{\mathcal{A}}$ and $\widehat{\Sigma}_\Lambda = \widehat{\Sigma}_{\mathcal{A}}$. Since W_Λ permutes the elements of $\mathcal{A}$, there is an induced action $W_\Lambda \curvearrowright M(\mathcal{A} \otimes \mathbb{C})$ and an induced action $W_\Lambda \curvearrowright \widehat{\Sigma}_{\mathcal{A}}$, which are free. The union of $\mathcal{A}$ cuts the unit sphere of $\mathbb{R}^n$ into a simplicial complex, which is isomorphic to the Coxeter complex $\mathfrak{C}_\Lambda$ and dual to Σ_Λ. The following are standard [Par14, §3.2 and 3.3].

Theorem 2.13. *Suppose $\mathcal{A}$ is the reflection arrangement associated with a finite Coxeter group W_Λ. Then*

- $\pi_1 M(\mathcal{A} \otimes \mathbb{C}) = PA_\Lambda$ [vdL83],
- $\pi_1(M(\mathcal{A} \otimes \mathbb{C})/W_\Lambda) = \pi_1(\widehat{\Sigma}_\Lambda/W_\Lambda) = A_\Lambda$,
- $\widehat{\Sigma}^2_\Lambda/W_\Lambda$ *is isomorphic to the presentation complex of* A_Λ.

Now we consider the special case when Λ is a linear graph with consecutive vertices $\{s_i\}_{i=1}^n$. We define a relation on the vertex set of Δ_Λ as follows. Let $x, y \in \Delta_\Lambda$ be vertices of type $\hat{s}_i$ and $\hat{s}_j$, respectively. We say $x < y$ if they are adjacent in Δ_Λ and $i < j$. A similar relation can be defined for the vertex set of the Coxeter complex $\mathfrak{C}_\Lambda$. This relation is actually transitive (see, e.g., [Hua23, Cor 6.5]), so we obtain a poset, which is graded and has rank n.

Recall that Λ is of type A_n if each edge is labeled by 3, in which case A_Λ is the braid group on $n+1$ stands, and W_Λ is the symmetric group on $n+1$ letters. The corresponding reflection arrangement is obtained by intersecting the hyperplanes $\{x_i = x_j\}_{1\le i<j\le n+1}$ of $\mathbb{R}^{n+1}$ with the subspace

$$V = \{(x_1, \dots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i = 0\} \cong \mathbb{R}^n.$$

Up to a linear transformation, we can represent an A_n-type reflection arrangement as the hyperplanes $\{x_i = 0\}_{i=1}^n$ and $\{x_i = x_j\}_{1\le i<j\le n}$ in $\mathbb{R}^n$. We will call this arrangement a *skewed A_n arrangement*, as it differs with the actual A_n arrangement by a linear transformation.

Theorem 2.14. *Suppose Λ is of type A_n. Then the posets $(\Delta^0_\Lambda, <)$ and $(\mathfrak{C}^0_\Lambda, <)$ are bowtie free.*

The Coxeter case follows from Tits's work [Tit74], see [Hir20, Thm 2.10] for an explanation. The Artin case is due to Crisp-McCammond (unpublished), see [Hae21, §5] for an explanation.

A Coxeter diagram Λ is of type B_n if Λ is linear graph with consecutive vertices $\{s_i\}_{i=1}^n$ such that all edges are labeled by 3 except the edge between s_{n-1} and s_n, which is labeled by 4. Then W_Λ is the finite Coxeter group of type B_n (sometimes called the "signed symmetric group"), with the associated reflection arrangement made up of $\{x_i = \pm x_j\}_{1\le i<j\le n}$ and $\{x_i = 0\}_{1\le i\le n}$ in $\mathbb{R}^n$. Instead of describing the Coxeter complex $\mathfrak{C}_\Lambda$ as the subdivision of the unit sphere of $\mathbb{R}^n$ by these hyperplanes, it is instructive to think of $\mathfrak{C}_\Lambda$ as the barycentric subdivision of the boundary of the cube $[-1,1]^n$ in $\mathbb{R}^n$. Then vertices of $\mathfrak{C}_\Lambda$ are in 1-1 correspondence with points in $\mathbb{R}^n$ whose each coordinate belongs to $\{-1,0,1\}$ with at least one nonzero coordinate. Such a point corresponds to a vertex of type $\hat{s}_i$ if and only if it has i nonzero coordinates. Moreover, if P is obtained by adding an extra minimal element to $(\mathfrak{C}^0_\Lambda, <)$, then the geometric realization $|P|$ is isomorphic to the barycentric subdivision of the entire cube $[-1,1]^n$, which is naturally made of unit orthoschemes, and $|P|_\infty$ is isometric to $[-1,1]^n$ with the ℓ^∞ metric.

Theorem 2.15. *Suppose Λ is of type B_n. Then $(\Delta^0_\Lambda, <)$ and $(\mathfrak{C}^0_\Lambda, <)$ are bowtie free and upward flag posets.*

The Coxeter case of the above theorem is straight forward. The Artin case is proved in [Hae21, Prop 6.6], which is deduced from Theorem 2.14.

The Coxeter diagram Λ of type D_n (for $n \ge 3$) is shown in Figure 1, where all edges are labeled by 3. The associated reflection arrangement is $\{x_i = \pm x_j\}_{1\le i<j\le n}$ in $\mathbb{R}^n$. We subdivide each edge of Δ_Λ connecting a vertex of type $\hat{s}_n$ and a vertex of type $\hat{s}_{n-1}$, and declare the middle point of such edge is of type m. Cut each top dimensional simplex in Δ_Λ into two simplices along the codimension 1 simplex spanned by vertices of type m and $\{\hat{s}_i\}_{i=1}^{n-2}$. This gives a new simplicial complex, denoted by Δ'_Λ. Define a map t from $(\Delta'_\Lambda)^0$ to $\{1,2,\ldots,n\}$ by sending vertices of type $\hat{s}_i$ to i for $1 \le i \le n-2$, vertices of type m to $n-1$, and vertices of type $\hat{s}_n$ and $\hat{s}_{n-1}$ to n. We define a relation $<$ on $(\Delta'_\Lambda)^0$ as follows. For two vertices x, y of Δ'_Λ, $x < y$ if x and y are adjacent in Δ'_Λ and $t(x) < t(y)$. The simplicial complex Δ'_Λ, together with this relation, is called the *(s_n, s_{n-1})-subdivision of Δ_Λ*. Similarly, we can define (s_n, s_{n-1})-subdivision $\mathfrak{C}'_\Lambda$ of $\mathfrak{C}_\Lambda$. Note that $\mathfrak{C}'_\Lambda$ is isomorphic to the Coxeter complex of type B_n via a type-preserving isomorphism. While Δ'_Λ is not isomophic to the Artin complex of type B_n, the following conjecture, due to Haettel, predicts that these two complexes share the following property – his original motivation for this conjecture is that it leads an alternative proof that Artin group of type $\widetilde{D}_n$ satisfies the $K(\pi,1)$-conjecture.

Conjecture 2.16. *(Haettel) Suppose Λ is of type D_n for $n \ge 3$. Then $((\Delta'_\Lambda)^0, <)$ is a poset that is bowtie free and upward flag.*

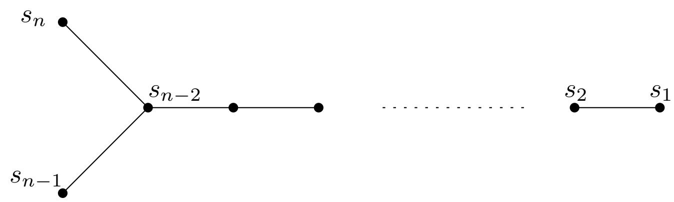

FIGURE 1. Coxeter diagram of type D_n.

The poset part of the conjecture is straightforward. The bowtie free part is a consequence of [Hua23, Cor 8.2]. The upward flag part reduces to checking properties of certain 6-cycles in the Artin complex Δ_Λ, see e.g. [Hua24b, Lem 7.6]. The following is proved in [Hua24b].

Theorem 2.17. *Conjecture 2.16 holds when $n = 3, 4$.*

3. THE FALK COMPLEXES AND THEIR METRIC

3.1. **Falk complexes.** In [Fal95], for each affine arrangement in $\mathbb{C}^2$ which is the complexification of a real arrangement, Falk described a locally infinite complex which is homotopy equivalent to the associated arrangement complement in $\mathbb{C}^2$. Falk showed that, compared to other complexes, this complex has the advantage of being easier to arrange a "non-positive curvature-like" structure (under some additional assumptions), and hence lead to contractibility results. Now we describe what we believe should be the correct analogue of Falk's complex in higher dimensions. Although it is no longer true that the complex is homotopy equivalent to the associated complement in $\mathbb{C}^n$, a weaker statement (see Theorem 3.3 below) holds, which is still useful for proving $K(\pi, 1)$ results.

Let $\mathcal{A}$ be an affine hyperplane arrangement of $\mathbb{R}^n$. Let $D_\mathcal{A}$ be the union of all elements in Fan($\mathcal{A}$) which are bounded in $\mathbb{R}^n$. Note that $D_\mathcal{A}$ is naturally a stratified space (in the sense of [BH99, Chapter II.12.1]) by considering its fans. We also call fans in $D_\mathcal{A}$ as *open cells* of $D_\mathcal{A}$, as $D_\mathcal{A}$ also has the structure of a polyhedral complex, with each fan being an open cell in this complex. A *face* of $D_\mathcal{A}$ is defined to be the closure of an open cell of $D_\mathcal{A}$. Each face of $D_\mathcal{A}$ is a disjoint union of fans. Let $\Sigma_\mathcal{A}$ be as in Section 2.1. A face F of $D_\mathcal{A}$ is *dual* to a face F' of $\Sigma_\mathcal{A}$ if the barycenter of F' is contained in the interior of F. In this case, we will also say $F \subseteq D_\mathcal{A}$ is dual to the standard subcomplex $\widehat{F}'$ of $\widehat{\Sigma}_\mathcal{A}$.

Definition 3.1. Let $\widetilde{K}$ be the universal cover of $\widehat{\Sigma}_\mathcal{A}$. For each face F of $D_\mathcal{A}$, let F' be the face of $\Sigma_\mathcal{A}$ dual to F. We index the collection of elevations of $\widehat{F}'$ in $\widetilde{K}$ using an index set Λ_F (recall that an *elevation* of $\widehat{F}'$ in $\widetilde{K}$ is a connected component of the inverse image of $\widehat{F}'$ with respect to the map $\widetilde{K} \to \widehat{\Sigma}_\mathcal{A}$). Let $\mathcal{F}$ be the collection of faces of $D_\mathcal{A}$. Then we define the *Falk complex* of $\mathcal{A}$ to be

$$\mathbb{F}_\mathcal{A} = \left(\bigsqcup_{F \in \mathcal{F}} F \times \Lambda_F \right) / \sim,$$

where we identify $F_1 \times \{\lambda_1\}$ as a face of $F_2 \times \{\lambda_2\}$ ($F_1, F_2 \in \mathcal{F}$, $\lambda_1 \in \Lambda_{F_1}$ and $\lambda_2 \in \Lambda_{F_2}$) if $F_1 \subseteq F_2$ in $D_\mathcal{A}$ and the elevation of $\widehat{F}'_2$ in $\widetilde{K}$ associated with λ_2 is contained in the elevation of $\widehat{F}'_1$ associated with λ_1. The action of $\pi_1 \widehat{\Sigma}_\mathcal{A}$ on $\widetilde{K}$ by deck transformations induces an action $\pi_1\widehat{\Sigma}_{\mathcal{A}} \curvearrowright \mathbb{F}_{\mathcal{A}}$, whose quotient is naturally identified with $D_{\mathcal{A}}$. This gives a map $q : \mathbb{F}_{\mathcal{A}} \to D_{\mathcal{A}}$.

Lemma 3.2. *(1) Any elevation of a standard subcomplex $\widehat{E}$ of $\widehat{\Sigma}_{\mathcal{A}}$ is a copy of the universal cover of $\widehat{E}$.*

(2) Let $\widehat{E}_1$ and $\widehat{E}_2$ be two standard subcomplexes of $\widehat{\Sigma}_{\mathcal{A}}$. For $i = 1, 2$, let $\widetilde{E}_i$ be an elevation of $\widehat{E}_i$ in $\widetilde{K}$. If $\widetilde{E}_1 \cap \widetilde{E}_2 \neq \emptyset$, then this intersection is an elevation of $\widehat{E}_1 \cap \widehat{E}_2$.

Proof. By Definition 2.4, there is retraction from $\widehat{\Sigma}_{\mathcal{A}}$ to $\widehat{E}$, so $\hat{E} \to \widehat{\Sigma}_{\mathcal{A}}$ is π_1-injective. Thus (1) follows. The proof of (2) is similar to [Hua24a, Lemma 6.3], and we provide details here for the convenience of the reader. For (2), it suffices to prove $\widetilde{E}_1 \cap \widetilde{E}_2$ is connected. Given vertices $x, y \in \widetilde{E}_1 \cap \widetilde{E}_2$, let $\widetilde{P}_i$ be a path in $\widetilde{E}_i$ from x to y for $i = 1, 2$. Let P_i be the path which is the image of $\widetilde{P}_i$ in $\widehat{E}_i$ under the covering map. Then P_1 and P_2 are homotopic rel endpoints in $\widehat{\Sigma}_{\mathcal{A}}$. Let $\Pi_{\widehat{E}_1} : \widehat{\Sigma}_{\mathcal{A}} \to \widehat{E}_1$ be the retraction map in Definition 3.1 and $Q_i = \Pi_{\widehat{E}_1}(P_i)$. Then $Q_1 = P_1$ and $Q_2 \subseteq \widehat{E}_1 \cap \widehat{E}_2$ by Lemma 2.5. As Q_1 and Q_2 are homotopic rel endpoints in $\widehat{\Sigma}_{\mathcal{A}}$, we know Q_2 and P_2 are homotopic rel endpoints in $\widehat{\Sigma}_{\mathcal{A}}$. Hence Q_2 lifts to a path in $\widetilde{E}_1 \cap \widetilde{E}_2$ connecting x and y, as desired. □

Given an affine hyperplane arrangement $\mathcal{A}$ of $\mathbb{R}^n$, and a point $x \in \mathbb{R}^n$, the *local arrangement* at x, denoted by $\mathcal{A}_x$, is made of the collection of all hyperplanes of $\mathcal{A}$ that contain x. A point $x \in \mathbb{R}^n$ is an *$\mathcal{A}$-vertex*, if it is an element of Fan($\mathcal{A}$) (or equivalently, it is a vertex of $D_{\mathcal{A}}$).

Theorem 3.3. *Let $\mathcal{A}$ be an affine hyperplane arrangement of $\mathbb{R}^n$. Then its Falk complex $\mathbb{F}_{\mathcal{A}}$ is simply connected.*

Suppose in addition that for all $\mathcal{A}$-vertices of $\mathbb{R}^n$, the local arrangements at these vertices are $K(\pi, 1)$. If the Falk complex $\mathbb{F}_{\mathcal{A}}$ of $\mathcal{A}$ is contractible, then $\mathcal{A}$ is $K(\pi, 1)$.

Proof. Let $\{x_i\}_{i\in I}$ be the collection of $\mathcal{A}$-vertices of $\mathbb{R}^n$. Let $\widehat{E}_i$ be the standard subcomplex of $\widehat{\Sigma}_{\mathcal{A}}$ dual to x_i. Then $\widehat{E}_i$ is the Salvetti complex for the local arrangement $\mathcal{A}_{x_i}$. Note that $\{\widehat{E}_i\}_{i\in I}$ forms a covering of $\widehat{\Sigma}_{\mathcal{A}}$. Now we consider the covering $\mathcal{U}$ of $\widetilde{K}$ by all possible elevations of elements in $\{\widehat{E}_i\}_{i\in I}$. Let Δ be the nerve of $\mathcal{U}$. By Lemma 3.2, each member of $\mathcal{U}$ is simply connected, and the intersection of finitely many members in $\mathcal{U}$ is also simply connected (when non-empty). Then by [Bjö03, Theorem 6], Δ is simply connected.

Elements of $\mathcal{U}$ are in 1-1 correspondence with vertices of $\mathbb{F}_{\mathcal{A}}$. Moreover, a finite collection of elements in $\mathcal{U}$ has non-empty common intersection if and only if the associated collection of vertices in $\mathbb{F}_{\mathcal{A}}$ have non-empty common intersection of their open stars. Let $\mathcal{U}'$ be the covering of $\mathbb{F}_{\mathcal{A}}$ by open stars of vertices of $\mathbb{F}_{\mathcal{A}}$. Then the nerve Δ' of $\mathcal{U}'$ is isomorphic to Δ. Moreover, the intersection of finitely many members in $\mathcal{U}'$ is contractible (if non-empty). Thus by [Bjö03, Theorem 6] Δ' is homotopy equivalent to $\mathbb{F}_{\mathcal{A}}$. Therefore $\mathbb{F}_{\mathcal{A}}$ is simply connected.

If the local arrangements at vertices of $D_{\mathcal{A}}$ are $K(\pi, 1)$, then $\widehat{E}_i$ is aspherical. Moreover, any standard subcomplex of $\widehat{E}_i$ is also aspherical, as it is a retract of $\widehat{E}_i$ by Definition 2.4. Thus any finite intersection of elements in $\mathcal{U}$ is contractible (if non-empty). Hence [Bjö03, Theorem 6] implies that Δ is homotopy equivalent to $\widetilde{K}$. Thus if $\mathbb{F}_{\mathcal{A}}$ is contractible, then $\widetilde{K}$ is contractible. □

3.2. **Links of Falk complexes.** When $\mathcal{A}$ is a central arrangement, there is another complex associated with $\mathcal{A}$, called the *spherical Deligne complex*, defined as follows. Let $S_{\mathcal{A}}$ be the unit sphere, endowed with the polyhedral complex structure coming from the the intersection of the unit sphere of $\mathbb{R}^n$ with $\mathcal{A}$. Then there is a 1-1 correspondence between open cells in $S_{\mathcal{A}}$ and elements in $\mathrm{Fan}(\mathcal{A})$ which are not 0-dimensional. A *face* of $S_{\mathcal{A}}$ is the closure of an open cell of $S_{\mathcal{A}}$. Each face is a disjoint union of open cells. A face F of $S_{\mathcal{A}}$ is dual to a face F' of $\Sigma_{\mathcal{A}}$ if the barycenter of F' is contained in the fan associated with F. In this case, we also say F is dual to the standard subcomplex $\widehat{F}'$ of $\widehat{\Sigma}_{\mathcal{A}}$. The definition of the spherical Deligne complex of $\mathcal{A}$, denoted by $\Delta_{\mathcal{A}}$, is identical to Definition 3.1, except we consider faces of $S_{\mathcal{A}}$ instead of faces of $D_{\mathcal{A}}$. Similarly, we have a natural action $\pi_1\widehat{\Sigma}_{\mathcal{A}} \curvearrowright \Delta_{\mathcal{A}}$ which induces a map $\Delta_{\mathcal{A}} \to S_{\mathcal{A}}$.

Lemma 3.4. *Suppose $\mathcal{A}$ is the reflection arrangement associated with a finite Coxeter group with Coxeter diagram Λ. Then $\Delta_{\mathcal{A}}$ is isomorphic to the Artin complex Δ_Λ.*

Proof. By Theorem 2.13, the 1-skeleton of the universal cover of $\widehat{\Sigma}_{\mathcal{A}}$ can be identified with the Cayley graph of A_Λ. As $\widehat{\Sigma}_{\mathcal{A}}$ is dual to the Coxeter complex $\mathfrak{C}_\Lambda$, and the 1-skeleton of $\widehat{\Sigma}_{\mathcal{A}}$ is obtained from the 1-skeleton of $\widehat{\Sigma}_{\mathcal{A}}$ by replacing each edge by a pair of edges, we deduce that the 1-skeleton $\widehat{\Sigma}_{\mathcal{A}}$ is naturally identified with the Cayley graph of W_Λ. Hence each edge of $\widehat{\Sigma}_{\mathcal{A}}$ is labeled by a generator of W_Λ, i.e. a vertex of Λ. Let S be the collection of vertices of Λ. Given a face F' of $\Sigma_{\mathcal{A}}$ with associated standard subcomplex $\widehat{F}'$ of $\widehat{\Sigma}_{\mathcal{A}}$, let T be the set of labels of edges in $\widehat{F}'$. By Definition 2.4, $\widehat{F}' \to \widehat{\Sigma}_{\mathcal{A}}$ is π_1-injective. Thus elevations of $\widehat{F}'$ in the universal cover of $\widehat{\Sigma}_{\mathcal{A}}$ are in 1-1 correspondence with left cosets of A_T (i.e. the standard parabolic subgroup of A_Λ generated by T) in A_Λ.

On the other hand, we define the *type* of a simplex in the Artin complex Δ_Λ to be the intersection of the types of its vertices. Thus there is a 1-1 correspondence between simplices in Δ_Λ of type T and left cosets of A_T in Δ_Λ. Now the lemma follows from the definition of Δ_Λ and $\Delta_{\mathcal{A}}$. □

Let $\mathcal{A}$ be an arbitrary affine hyperplane arrangement. Let x be a vertex in $D_{\mathcal{A}}$, with $\mathcal{A}_x$ be the local arrangement at x. Then $\mathrm{lk}(x, D_{\mathcal{A}})$ can be naturally identified with a subcomplex of $S_{\mathcal{A}_x}$ as in Lemma 3.5, which is a consequence of the description of $\mathbb{F}_{\mathcal{A}}$ and $\Delta_{\mathcal{A}_x}$.

Lemma 3.5. *(*[Hua24a, Lemma 4.10]*) Let $x' \in \mathbb{F}_{\mathcal{A}}$ be a vertex which maps to $x \in D_{\mathcal{A}}$ under $\mathbb{F}_{\mathcal{A}} \to D_{\mathcal{A}}$. Let N be the inverse image of* $\mathrm{lk}(x, D_{\mathcal{A}})$ *(viewed as a subset of $S_{\mathcal{A}_x}$) under the map $\Delta_{\mathcal{A}_x} \to S_{\mathcal{A}_x}$. Then* $\mathrm{lk}(x', \mathbb{F}_{\mathcal{A}}) \cong N$.

3.3. **Metrizing Falk complexes.**

Definition 3.6. Suppose $\mathcal{A}$ is an admissible affine arrangement in $\mathbb{R}^n$, and let $\mathbb{F}_{\mathcal{A}}$ be the associated Falk complex. We define a metric d_∞ on $\mathbb{F}_{\mathcal{A}}$ as follows. Let $D_{\mathcal{A}}$ be defined as in the previous section. As $D_{\mathcal{A}}$ is a subset of $\mathbb{R}^n$, we endow $D_{\mathcal{A}}$ with the ℓ^∞ metric d_∞ on $\mathbb{R}^n$. This restricts to a metric on each face of $D_{\mathcal{A}}$, hence an ℓ^∞-metric on each face of $\mathbb{F}_{\mathcal{A}}$ using the map $\mathbb{F}_{\mathcal{A}} \to D_{\mathcal{A}}$. Then we can define a pseudo-metric d_∞ on $\mathbb{F}_{\mathcal{A}}$ by considering the infimum of lengths of strings between each pair of points as in Section 2.3.

Lemma 3.7. *The pseudometric d_∞ is a metric. Moreover, $(\mathbb{F}_\mathcal{A}, d_\infty)$ is a complete geodesic metric space.*

Proof. Note that if $\mathbb{F}_\mathcal{A}$ has only finitely many isometry types of its closed cells, then the lemma follows from the same argument in [BH99, p. 101, Theorem I.7.13 and p. 105, Theorem I.7.19]. In the more general case, note that the map $(\mathbb{F}_\mathcal{A}, d_\infty) \to (D_\mathcal{A}, d_\infty)$ is 1-Lipschitz. So for any $x \in \mathbb{F}_\mathcal{A}$, the ball $B(x, r)$ in $(\mathbb{F}_\mathcal{A}, d_\infty)$ is mapped to a bounded region in $(D_\mathcal{A}, d_\infty)$. So the smallest subcomplex of $\mathbb{F}_\mathcal{A}$ containing $B(x, r)$ has only finitely many isometry types of its closed cells. As the lemma only concerns properties that need to be verified on each ball, this finishes the proof. □

3.4. **Falk complexes for admissible arrangements.** For $i = 1, 2$, let $\mathcal{A}_i$ be an affine hyperplane arrangement in $\mathbb{R}^{m_i}$. Then the *product* of $\mathcal{A}_1$ and $\mathcal{A}_2$ is an arrangement $\mathcal{A}$ in $\mathbb{R}^{m_1+m_2}$ whose hyperplanes are of form $H \times \mathbb{R}^{m_2}$ for $H \in \mathcal{A}_1$ or $\mathbb{R}^{m_1} \times H$ for $H \in \mathcal{A}_2$.

Definition 3.8. We say an affine hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is *admissible*, if at each $\mathcal{A}$-vertex x, the local arrangement at x is a translate of one of the following four types:

(1) (type B_n) $x_i \pm x_j = 0$ for $1 \leq i \neq j \leq n$ and $x_i = 0$ for $1 \leq i \leq n$;
(2) (type D_n) $x_i \pm x_j = 0$ for $1 \leq i \neq j \leq n$;
(3) (skewed type A_n) $x_i = 0$ for $1 \leq i \leq n$ and $x_i = x_j$ for $1 \leq i \neq j \leq n$, or any image of this this arrangement under the $(\mathbb{Z}/2\mathbb{Z})^n$ action on $\mathbb{R}^n$ by reflections about the coordinate hyperplanes;
(4) or a product of the previous types.

Note that a skewed type A_1 arrangement is just the hyperplane $x = 0$ in $\mathbb{R}$.

For $1 \leq i \leq 4$, let $\mathcal{A}(i)$ be the central arrangement in item (i) of the above list. Let $S_{\mathcal{A}(i)}$ and $\Delta_{\mathcal{A}(i)}$ be defined in Section 3.2. Note that for $i > 1$, we can subdivide $S_{\mathcal{A}(i)}$ into $S'_{\mathcal{A}(i)} := S_{\mathcal{A}(1)}$ since $\mathcal{A}(i) \subseteq \mathcal{A}(1)$. Let $\Delta'_{\mathcal{A}(i)}$ be the subdivision of $\Delta_{\mathcal{A}(i)}$ obtained from pulling back the subdivision $S'_{\mathcal{A}(i)}$ of $S_{\mathcal{A}(i)}$ via the map $\Delta_{\mathcal{A}(i)} \to S_{\mathcal{A}(i)}$. Let $\Delta'_{\mathcal{A}(1)} := \Delta_{\mathcal{A}(1)}$.

Definition 3.9. Let $1 \leq i \leq 3$. The *u-type* (short for "unsubdivided type") of a vertex in $S_{\mathcal{A}(i)}$ is the usual type of the vertex viewed as a Coxeter complex as labeled in Section 2.4. For $1 \leq i \leq 4$, the *s-type* ("subdivided type") of a vertex in $S'_{\mathcal{A}(i)}$ is the type of the corresponding vertex in $S_{\mathcal{A}(1)}$ viewed as the Coxeter complex $\mathfrak{C}_\Lambda$ of type $\Lambda = B_n$, also as labeled in Section 2.4. The u-type of a vertex of $\Delta_{\mathcal{A}(i)}$ is its usual type from the spherical Deligne complex, and the s-type of a vertex in $\Delta'_{\mathcal{A}(i)}$ is defined to be the pull-back of s-type of vertices in $S'_{\mathcal{A}(i)}$.

We define a relation $<$ on $(\Delta'_{\mathcal{A}(i)})^0$. Given two distinct vertices $v, w \in (\Delta'_{\mathcal{A}(i)})^0$ of type $\hat{s}_i, \hat{s}_j$ respectively, define $v < w$ if v and w are adjacent in $\Delta'_{\mathcal{A}(i)}$ and $i < j$. As is common convention, we say $v \leq w$ if $v < w$ or $v = w$. We call this the *s-order* and sometimes write $\leq_s$. We call the usual ordering on $\Delta^0_{\mathcal{A}(i)}$ the *u-order* and sometimes write $\leq_u$.

Lemma 3.10. *For $1 \leq i \leq 3$ the s-order $\leq_s$ on $\Delta'_{\mathcal{A}(i)}$ is a partial order.*

Proof. The case $i = 1$ is verified in [Hae21, Lem 6.5], the case $i = 2$ is verified in [Hua24a, Lem 2.22], and the case $i = 3$ is verified in Proposition 5.15. □

For the type (4) arrangements, we have a nice description of the relation $\leq$.

Lemma 3.11. *Suppose $\mathcal{B} = \mathcal{B}_1 \times \cdots \times \mathcal{B}_k$, where each $\mathcal{B}_j$ is an arrangement of type (1), (2), or (3). For each i, let $(\mathcal{P}_i, <)$ denote $((\Delta'_{\mathcal{B}_i})^0, <)$ with a minimal element 0 added. Then $((\Delta'_{\mathcal{B}})^0, <)$ is isomorphic to $(\mathcal{P}_1 \times \cdots \times \mathcal{P}_k \setminus (0, \ldots, 0), <)$ with the product relation defined by $(p_1 \ldots, p_n) < (p'_1, \ldots, p'_n)$ if $p_i < p'_i$ for each i.*

In particular, the s-order on $\Delta'_{\mathcal{A}(4)}$ is a partial order.

Proof. For each j, we know $\mathcal{P}_j$ is a poset by Lemma 3.10. The geometric realization $|\mathcal{P}_i|$ of $\mathcal{P}_i$ is a cone over $\Delta'_{\mathcal{B}_i}$. Let $\mathcal{P} = \mathcal{P}_i \times \cdots \times \mathcal{P}_k$. Then $\mathcal{P}$ is a poset (under the product order) whose geometric realization $|\mathcal{P}|$ is a product $|\mathcal{P}_1| \times \cdots \times |\mathcal{P}_k|$ of spaces. On the other hand, as $\mathcal{B}$ is a product of the $\mathcal{B}_i$, we know the cone over $\Delta'_{\mathcal{B}}$ is a product of the cones over the $\Delta'_{\mathcal{B}_i}$. Thus $((\Delta'_{\mathcal{B}})^0, <)$ is isomorphic to $\mathcal{P}$ with the minimal element $(0, \ldots, 0)$ removed, which is still a poset. □

We can summarize these lemmas as

Proposition 3.12. *For $1 \leq i \leq 4$ the s-order $\leq$ on $\Delta'_{\mathcal{A}(i)}$ is a partial order.*

Lemma 3.13. *For $1 \leq i \leq 4$, a collection of vertices of $\Delta'_{\mathcal{A}(i)}$ span a simplex of $\Delta'_{\mathcal{A}(i)}$ if and only if they correspond to a chain in $((\Delta'_{\mathcal{A}(i)})^0, <)$.*

Proof. The forward implication follows from Proposition 3.12; namely, if V is a set of vertices which span a simplex, then they are pairwise adjacent, hence pairwise comparable, and thus form a chain since $\leq$ is a partial order. The reverse implication will follow from the fact that each subdivided complex is flag. Assuming this, if V is a chain, then the elements are pairwise comparable, hence pairwise adjacent, and hence span a simplex. We now see why these are flag complexes. For $i = 1$, since the subdivision is trivial, this follows from the general fact for Artin complexes [GP12, Prop. 4.5]. For $i = 2$, this follows easily from the description of the subdivided complex for D_n type given in Section 2.4. For $i = 3$, this is Proposition 5.16. The case $i = 4$ follows from (the proof of) Lemma 3.11: the cone over $\Delta'_{\mathcal{B}}$ is a product of the cones over the $\Delta'_{\mathcal{B}_i}$, the cone over a flag complex is flag, the product of flag complexes is flag, and the link of a vertex in a flag complex is flag. Since the link of the cone point is isomorphic to $\Delta'_{\mathcal{B}}$, the result follows. □

An affine hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is *complete* if $D_{\mathcal{A}} = \mathbb{R}^n$. We say $\mathcal{A}$ has *finite shape*, if there are only finitely many isometry types of cells in $D_{\mathcal{A}}$. Note that if $\mathcal{A}$ is invariant under the action of a translation group $\mathbb{Z}^n$, then $\mathcal{A}$ is both complete and has finite shape.

Lemma 3.14. *Suppose $\mathcal{A}$ is a complete admissible affine arrangement in $\mathbb{R}^n$ with finite shape. Suppose for $1 \leq i \leq 4$, the poset $((\Delta'_{\mathcal{A}(i)})^0, <)$ is bowtie free and upward flag. Then $(\mathbb{F}_{\mathcal{A}}, d_\infty)$ is an injective metric space. In particular, $\mathbb{F}_{\mathcal{A}}$ is contractible.*

Proof. Let $\pi : \mathbb{F}_{\mathcal{A}} \to D_{\mathcal{A}}$ be the 1-Lipschitz projection map. Given $x \in \mathbb{F}_{\mathcal{A}}$ mapping to an $\mathcal{A}$-vertex $\bar{x} \in D_{\mathcal{A}}$, let $\varepsilon(x)$ be a positive number such that the ball $B(\bar{x}, \varepsilon(x))$ in $D_{\mathcal{A}}$ is contained in the open star of $\bar{x}$ in $D_{\mathcal{A}}$ (here the ball is taken with respect to the ℓ^∞-metric). As π is 1-Lipschitz, we know $B(x, \varepsilon(x))$ is the connected component of $\pi^{-1}(B(\bar{x}, \varepsilon(x)))$ that contains x. Suppose the local arrangement at $\bar{x}$ is $\mathcal{A}(i)$ (up to a translation). Then the subdivision of $S_{\mathcal{A}(i)}$ described as above induces a subdivision of $B(\bar{x}, \varepsilon(x))$ (which is a cube with side length $2\varepsilon(x)$) into orthoschemes of size $\varepsilon(x)$. Hence $B(x, \varepsilon(x))$ is also subdivided into a simplicial complex made up of orthoschemes of size $\varepsilon(x)$. Let $\mathcal{P}$ be the poset obtained by adding a minimal element to $((\Delta'_{\mathcal{A}(i)})^0, <)$. Then by Lemma 3.5 and Lemma 3.13, the simplicial structure on $B(x, \varepsilon(x))$ is isomorphic to the geometric realization $|\mathcal{P}|$ of $\mathcal{P}$. Such an isomorphism induces a natural bijection f from $B(x, \varepsilon(x))$ to the $\varepsilon(x)$-ball around the cone point in $|\mathcal{P}|_\infty$. Note that if $\varepsilon(x)$ is small enough, then in order to measure $d_\infty(y_1, y_2)$ for $y_1, y_2 \in B(x, \varepsilon(x))$, we only need to consider strings from y_1, y_2 that are contained in the open star of x in $\mathbb{F}_\mathcal{A}$, as strings which escape the open star will automatically have length $\geq 2\varepsilon(x) \geq d_\infty(y_1, y_2)$. Thus for such a choice of $\varepsilon(x)$, f is an isometry. By our assumptions and Theorem 2.10, $|\mathcal{P}|_\infty$ is an injective metric space, hence so are any of its balls, and in particular $B(x, \varepsilon(x))$ is an injective metric space.

We claim for each open face $F \subseteq \mathbb{F}_\mathcal{A}$, there is an $\varepsilon > 0$, depending only on the dimension of F, such that for each $x \in F$, $B(x, \varepsilon)$ is injective. We induct on the dimension of F. The case where F is a vertex is treated in the previous paragraph. Now suppose $\dim(F) = n > 0$. As ∂F is a disjoint union of finitely many open faces of lower dimension, by induction there is an $\varepsilon > 0$ such that $B(x, \varepsilon)$ is injective for each $x \in \partial F$. Thus for any $x \in F$ such that $d(x, \partial F) \leq \varepsilon/4$, $B(x, \varepsilon/4)$ is injective as $B(x, \varepsilon/4) \subseteq B(y, \varepsilon)$ for some $y \in \partial F$, and any ball in an injective metric space is injective. If $x \in F$ is such that $d(x, \partial F) \geq \varepsilon/4$, by a similar consideration of the position of strings as in the previous paragraph, there is an $\varepsilon' > 0$ such that the $B(x, \varepsilon')$ are isometric to each other for all such x. Thus we can assume $d(x, \partial F) = \varepsilon/4$ and finish as before. Given two different open faces F_1 and F_2 of $D_\mathcal{A}$, we write $F_1 \sim F_2$ if F_1 and F_2 are isometric, and each hyperplane of $\mathcal{A}$ containing F_1 is parallel to a hyperplane of $\mathcal{A}$ containing F_2 and vice versa. The relation $\sim$ is transitive and divides the collection of n-dimensional open faces of $D_\mathcal{A}$ into finitely many equivalent classes by the finite shape assumption. Moreover, if $F_1 \sim F_2$, $x_1 \in F_1$, and $x_2 \in F_2$, then a small ball around x_1 and a small ball around x_2 are isometric. Thus we can use the same ε for n-dimensional open faces in the same equivalence class. This proves the claim.

It follows that $\mathbb{F}_\mathcal{A}$ is locally uniformly injective. As $\mathbb{F}_\mathcal{A}$ is simply connected by Theorem 3.3, we deduce from Theorem 2.6 that $\mathbb{F}_\mathcal{A}$ is injective, hence contractible. □

We postpone the proof of the following key proposition to Section 5.

Proposition 3.15. *The poset $((\Delta'_{\mathcal{A}(3)})^0, <)$ is bowtie free and upward flag.*

Remark 3.16. There is an interesting contrast between this proposition and Theorem 2.14, as the poset in Theorem 2.14 is not upward flag.

Theorem 3.17. *Suppose $\mathcal{A}$ is a complete admissible affine arrangement in $\mathbb{R}^n$ with finite shape. Suppose Conjecture 2.16 holds. Then $(\mathbb{F}_\mathcal{A}, d_\infty)$ is an injective metric space and $\mathcal{A}$ is a $K(\pi, 1)$ arrangement.*

Proof. Proposition 3.15, Conjecture 2.16 and Theorem 2.15 imply that $((\Delta'_{\mathcal{A}(i)})^0, <)$ is bowtie free and upward flag for $i = 1$, 2, and 3, respectively. The bowtie free and upward flag properties are not changed by adding or deleting a minimal element, or by taking products. So by Lemma 3.11, $((\Delta'_{\mathcal{A}(i)})^0, <)$ is bowtie free and upward flag when $i = 4$. Then the theorem follows from Lemma 3.14 and Theorem 3.1. □

Corollary 3.18. *Suppose $\mathcal{A}$ is a complete finite shape admissible affine arrangement in $\mathbb{R}^n$ with $n = 2, 3, 4$, then $\mathcal{A}$ is a $K(\pi, 1)$ arrangement.*

Proof. This follows from Theorem 3.17 and Theorem 2.17. □

The next result can be proved in the same way as Theorem 3.17, using Proposition 3.15 and Theorem 2.15.

Corollary 3.19. *Suppose $\mathcal{A}$ is a complete finite shape affine arrangement in $\mathbb{R}^n$ such that for each $\mathcal{A}$-vertex x, the local arrangement at x is a translate of one of the following three types:*

(1) (type B_n) $x_i \pm x_j = 0$ for $1 \le i \ne j \le n$ and $x_i = 0$ for $1 \le i \le n$;
(2) (skewed type A_n) $x_i = 0$ for $1 \le i \le n$ and $x_i = x_j$ for $1 \le i \ne j \le n$, or any image of this this arrangement under the $(\mathbb{Z}/2\mathbb{Z})^n$ action on $\mathbb{R}^n$ by reflections about the coordinate hyperplanes;
(3) or a product of the previous types.

Then $(\mathbb{F}_{\mathcal{A}}, d_\infty)$ is an injective metric space and $\mathcal{A}$ is a $K(\pi, 1)$ arrangement.

4. Some examples of admissible arrangements

In each dimension, we consider an infinite family of finite affine arrangements, and an infinite family of infinite complete affine arrangements, as follows.

Definition 4.1. For $k \ge 1$, let $\mathcal{H}_{k,n}$ be the affine hyperplane arrangement in $\mathbb{R}^n$ given by $x_i \in \{-2k-1, -2k+1, \ldots, -3, -1, 1, 3, \ldots, 2k-1, 2k+1\}$ for $1 \le i \le n$ and $x_i \pm x_j = 0$ for $1 \le i \ne j \le n$.

For $k \ge 1$, let $\mathcal{K}_{k,n}$ be the affine hyperplane arrangement in $\mathbb{R}^n$ given by $x_i \in \mathbb{Z}$ for $1 \le i \le n$, and $x_i + x_j \in 2k\mathbb{Z} + 1$, $x_i - x_j \in 2k\mathbb{Z}$ for $1 \le i \ne j \le n$.

Proposition 4.2. *Both $\mathcal{H}_{k,n}$ and $\mathcal{K}_{k,n}$ are admissible in the sense of Definition 3.8.*

Proof. We only treat the family $\mathcal{K}_{k,n}$, as the family $\mathcal{H}_{k,n}$ is similar and much simpler. An $(x_i, x_j)^+$ hyperplane of $\mathcal{H}_{k,n}$ is a hyperplane defined by $x_i + x_j \in 2k\mathbb{Z} + 1$, and an $(x_i, x_j)^-$ hyperplane of $\mathcal{H}_{k,n}$ is a hyperplane defined by $x_i - x_j \in 2k\mathbb{Z} + 1$. An (x_i, x_j) hyperplane is either an $(x_i, x_j)^+$ hyperplane or an $(x_i, x_j)^-$ hyperplane. Let θ be a $\mathcal{K}_{k,n}$-vertex.

<u>Case 1: all the coordinates of θ are integers.</u> Then the local arrangement at θ contains all the coordinate hyperplanes. We define $x_i \sim x_j$ if there is an (x_i, x_j) hyperplane containing θ. We claim $\sim$ is an equivalence relation. Suppose $x_i - x_j = 2kn_1$ and $x_j - x_k = 2kn_2$ contain θ. Then $x_i - x_k = 2k(n_1 + n_2)$ contains θ. Suppose $x_i - x_j = 2kn_1$ and $x_j + x_k = 2kn_2 + 1$ contain θ. Then $x_i + x_k = 2k(n_1 + n_2) + 1$ contains θ. Suppose $x_i + x_j = 2kn_1 + 1$ and $x_j + x_k = 2kn_2 + 1$ contain θ. Then $x_i - x_k = 2k(n_2 - n_1)$ contains θ. Thus the claim is proved. As θ has integer coordinates, for each $i \ne j$, θ is not contained simultaneously in an $(x_i, x_j)^-$ hyperplane and an $(x_i, x_j)^+$ hyperplane. Thus the local arrangement at θ is a product of skewed type A arrangements, one for each equivalent class.

<u>Case 2: all the coordinates of θ are not integers</u>. Then the coordinates are of form $1/2 + k\mathbb{Z}$. We say the x_i-coordinate of θ is *even*, if it is $1/2 + k \cdot even$. Otherwise the x_i-coordinate of θ is *odd*. For $i \ne j$, if the x_i-coordinate and x_j-coordinate of θ have the same parity (i.e. they are both odd or both even), then we have both an $(x_i, x_j)^-$ hyperplane and an $(x_i, x_j)^+$ hyperplane containing θ. If the x_i-coordinate and x_j-coordinate of θ have different parity, then there are no $(x_i, x_j)^-$ hyperplanes and no $(x_i, x_j)^+$ hyperplanes containing θ. Thus the local arrangement at θ is a

product of two arrangements of type D, one involving the even coordinates, and another one involving the odd coordinates.

Case 3: the coordinates of θ has both integers and non-integers. Note that if x_i is an integer coordinate, and x_j is a non-integer coordinate, then there are no $(x_i, x_j)^-$ hyperplanes and no $(x_i, x_j)^+$ hyperplanes containing θ. Thus the local arrangement at θ is a product of the arrangements in Case 1 (between the integer coordinates) and the arrangements in Case 2 (between the non-integer coordinates). $\square$

Remark 4.3. Both $\mathcal{H}_{k,n}$ and $\mathcal{K}_{k,n}$ are not of fiber type in an obvious way. The family $\mathcal{K}_{k,n}$ generalizes [Fal95, Example 3.13] which is neither supersolvable nor simplicial. For $\mathcal{H}_{k,n}$, note that for each parallel family of hyperplanes in $\mathcal{H}_{k,n}$, one can find $\mathcal{H}_{k,n}$-vertices that are not contained in any member of this parallel family. So one cannot produce an iterated fibration structure on $\mathcal{H}_{k,n}$ in a similar as dealing with supersolvable arrangement complements.

To show $\mathcal{H}_{k,n}$ and $\mathcal{K}_{k,n}$ are $K(\pi,1)$ arrangements, we need the following which is a consequence of [Hua24a, Lem 5.5] and Theorem 2.15.

Proposition 4.4. *Let $\mathcal{A}(3)$, $S'_{\mathcal{A}(3)}$ and $\Delta'_{\mathcal{A}(3)}$ be as in Section 3.4. Let $(S'_{\mathcal{A}(3)})^+$ be the part of the sphere $S'_{\mathcal{A}(3)}$ in the first octant (i.e. all coordinates are ≥ 0). Let $(\Delta'_{\mathcal{A}(3)})^+$ be the inverse image of $(S'_{\mathcal{A}(3)})^+$ under $\Delta'_{\mathcal{A}(3)} \to S'_{\mathcal{A}(3)}$. Then the restriction of the partial order $(\Delta'_{\mathcal{A}(3)})^0$ to the vertex set of $(\Delta'_{\mathcal{A}(3)})^+$ is bowtie free and upward flag.*

Theorem 4.5. *Suppose Conjecture 2.16 holds in dimension n. Then $(\mathbb{F}_{\mathcal{H}_{k,n}}, d_\infty)$ and $(\mathbb{F}_{\mathcal{K}_{k,n}}, d_\infty)$ are injective metric spaces, and $\mathcal{H}_{k,n}$ and $\mathcal{K}_{k,n}$ are $K(\pi,1)$ arrangements for any $k \geq 1$.*

Thus by Theorem 2.17, for $n = 2, 3, 4$ and any $k \geq 1$, the arrangements $\mathcal{H}_{k,n}$ and $\mathcal{K}_{k,n}$ are $K(\pi,1)$ arrangements.

Proof. The $\mathcal{K}_{k,n}$ case is a consequence of Theorem 3.17 and Proposition 4.2, as it is complete. The arrangement $\mathcal{H}_{k,n}$ is not complete, but we will explain how it is still a $K(\pi,1)$ arrangement using techniques similar to the proof for complete arrangements. Let D be the union of all elements in $\mathrm{Fan}(\mathcal{H}_{k,n})$ that are bounded. Then D is the cube bounded by $\{x_i = \pm(2k+1)\}_{1 \leq i \leq n}$. Let $\mathbb{F}$ be the associated Falk complex. We can prove $(\mathbb{F}, d_\infty)$ is injective in the same as Lemma 3.14, except we need to verify that for each $\mathcal{A}$-vertex $\bar{x}$ on the boundary of D and $x \in \mathbb{F}$ that is mapped to $\bar{x}$, $B(x,r)$ with the induced metric from $(\mathbb{F}, d_\infty)$ is injective. Note that the local arrangement $\mathcal{A}_{\bar{x}}$ at $\bar{x}$ is a product of one skewed type A_m arrangement $\mathcal{L}_1$ and a few skewed type A_1 arrangements $\{\mathcal{L}_i\}_{i=2}^m$. Thus $S_{\mathcal{A}_{\bar{x}}}$ admits a join decomposition

$$S_{\mathcal{A}_{\bar{x}}} = S_{\mathcal{L}_1} \circ S_{\mathcal{L}_2} \circ \cdots \circ S_{\mathcal{L}_m}.$$

Note that $S_{\mathcal{L}_i}$ is a copy of $\mathbb{S}^0$ for $i \geq 2$. Let $\bar{N} = \mathrm{lk}(\bar{x}, D)$. Then $\bar{N} = \bar{N}_1 \circ \cdots \circ \bar{N}_k$, where $\bar{N}_1$ is the subset of $S_{\mathcal{L}_1}$ in the first octant (up to reflections along the coordinate hyperplanes). For $i \geq 2$, either $\bar{N}_i = S_{\mathcal{L}_i}$ or $\bar{N}_i$ is a single point. Now by Theorem 2.10, Lemma 3.5, Proposition 4.4 and the argument in the proof Theorem 3.17 for multiple factors, we know $B(x,r)$ is injective for r small enough. $\square$

5. The skewed A_n arrangement

Our goal in this section is to prove Theorem 5.20 (= Proposition 3.15). We will begin with a more in-depth discussion of the different subdivisions $S_{\mathcal{A}(3)}$ and $S'_{\mathcal{A}(3)}$ of the sphere (introduced in Section 3.4) in the specific case of type (3), how these subdivisions of the sphere relate, and how their properties extend to the complexes $\Delta_{\mathcal{A}(3)}$ and $\Delta'_{\mathcal{A}(3)}$. For ease of notation, **in the rest of this section, let $\mathcal{B} = \mathcal{A}(3)$.**

5.1. **The Coxeter complex.** It becomes convenient to replace $S_\mathcal{B}$ and $S'_\mathcal{B}$ with their projections to the boundary of the n-cube $[-1,1]^n$, inheriting the cell structures of $S_\mathcal{B}$ and $S'_\mathcal{B}$, resp. (not the cell structure of the cube). Equivalently, we can redefine $S_\mathcal{B}$ and $S'_\mathcal{B}$ replacing the unit sphere by $\partial[-1,1]^n$. For $S'_\mathcal{B}$, this gives the barycentric subdivision of the usual cell structure on $\partial[-1,1]^n$.

We briefly recall the labeling of the vertices of $S_\mathcal{B}$ and $S'_\mathcal{B}$. The general definition is given in Definition 3.9. The u-type of a vertex v of $S_\mathcal{B}$ is of the form $\hat{u}_i$; we define $u(v) = i$. Similarly, the s-type of a vertex w of $S'_\mathcal{B}$ has the form $\hat{s}_j$; we define $s(w) = j$.

There are two orderings we consider: the u-order and the s-order. The u-order is defined on vertices of $S_\mathcal{B}$, where we say $v <_u w$ if $v \sim_u w$ (meaning v and w are adjacent in the *u*nsubdivided complex $S_\mathcal{B}$) and $u(v) < u(w)$. The s-order is defined on vertices of $S'_\mathcal{B}$, where we say $v <_s w$ if $v \sim_s w$ (meaning v and w are adjacent in the *s*ubdivided complex $S'_\mathcal{B}$) and $s(v) < s(w)$. As is common, we say $v \leq_u w$ if either $v = w$ or $v <_u w$, and $v \leq_s w$ if either $v = w$ or $v <_s w$. Sometimes we may write $v \simeq_u w$ to say $v = w$ or $v \sim_u w$, and may write $v \simeq_s w$ to say $v = w$ or $v \sim_s w$.

Notice that if v is a vertex of $S_\mathcal{B}$ then it is also a vertex of $S'_\mathcal{B}$, so it has a u-type and and s-type, and these types do not necessarily agree. In particular there may be vertices v and w with $v \leq_u w$ but $w \leq_s v$ (or even $v \not\sim_s w$, so the vertices are not comparable in the s-order). We clarify the connection in the following.

For a vertex $p = (p_1, \ldots, p_n)$ in $S'_\mathcal{B}$, let $\mathrm{pos}(p) = \{\, i : p_i > 0 \,\} = \{\, i : p_i = 1 \,\}$ and $\mathrm{neg}(p) = \{\, i : p_i < 0 \,\} = \{\, i : p_i = -1 \,\}$ (recall we are projecting to the cube). The following facts follow quickly from the definitions and are left as an exercise.

Proposition 5.1. *Let v be a vertex of $S'_\mathcal{B}$.*

- $s(v) = \#\mathrm{pos}(v) + \#\mathrm{neg}(v)$.
- *If w is another vertex of $S'_\mathcal{B}$, then v and w are adjacent if and only if either* $\mathrm{pos}(v) \subseteq \mathrm{pos}(w)$ *and* $\mathrm{neg}(v) \subseteq \mathrm{neg}(w)$, *or* $\mathrm{pos}(w) \subseteq \mathrm{pos}(v)$ *and* $\mathrm{neg}(w) \subseteq \mathrm{neg}(v)$.
- *v is vertex of $S_\mathcal{B}$ if and only if either* $\mathrm{neg}(v) = \varnothing$ *or* $\mathrm{pos}(v) = \varnothing$. (In the first case, we will call v "non-negative", and in the second, we will call v "non-positive".)
- *If v is non-negative, then $u(v) = s(v) = \#\mathrm{pos}(v)$.*
- *If v is non-positive, then $u(v) + s(v) = n + 1$, or in other terms, $u(v) = n + 1 - s(v) = n + 1 - \#\mathrm{neg}(v)$.*

Note that there are always vertices (and generally, cells) in $S'_\mathcal{B}$ which are not vertices (resp., cells) of $S_\mathcal{B}$. We call the cells of $S_\mathcal{B}$ *real*, and the cells of $S'_\mathcal{B}$ which are not cells of $S_\mathcal{B}$ *fake*.

Adjacency in $S_\mathcal{B}$ is slightly more subtle than adjacency in $S'_\mathcal{B}$.

Lemma 5.2. *Let v and w be vertices of $S_{\mathcal{B}}$. If both v and w are non-negative or both non-positive, then they are adjacent in $S_{\mathcal{B}}$ if and only if they are adjacent in $S'_{\mathcal{B}}$. In this case, there is no fake vertex on the edge between them. If v is non-negative and w is non-positive, then they are adjacent in $S_{\mathcal{B}}$ if and only if their non-zero coordinates do not overlap. In this case, $u(v) < u(w)$, and there is a fake vertex b laying on the real edge between v and w satisfying $\mathrm{pos}(b) = \mathrm{pos}(v)$ and $\mathrm{neg}(b) = \mathrm{neg}(w)$.*

Proof. If both v and w are non-negative or both non-positive, then they are both in the same octant, and there is no subdivision within this octant, so adjacency in the unsubdivided complex is equivalent to adjacency in the subdivided complex.

Now let $v = (v_1, \ldots, v_n)$ be a non-negative vertex and $w = (w_1, \ldots, w_n)$ a non-positive vertex. Notice that as a basic consequence of the definition of $S_{\mathcal{B}}$, v and w are adjacent in $S_{\mathcal{B}}$ if and only if every hyperplane H containing neither v nor w has v and w contained in the same connected component of its complement $\mathbb{R}^n \backslash H$. This statement is equivalent to saying that v and w are vertices of the same chamber of the arrangement, which in turn is equivalent to adjacency because the arrangement is simplicial, and every edge is contained in a chamber.

If v and w have non-overlaping non-zero coordinates, then $v_i > 0$ implies $w_i = 0$ and $w_i < 0$ implies $v_i = 0$, hence v and w are contained in the same closed halfspace bounded by $\{x_i = 0\}$. Moreover, if $v_i > v_j$ for $i \neq j$ then $v_i > 0$ and $v_j = 0$. Consequently, $w_i = 0$, $w_j \leq 0$, and $w_i \geq w_j$. Similarly, $w_i > w_j$ implies $v_i \geq v_j$. So v and w are contained in the same closed halfspace bounded by $\{x_i = x_j\}$ for $i \neq j$. Hence v and w are contained in the same chamber, hence adjacent.

Conversely, suppose v and w are adjacent. Then for every hyperplane H, we either have that H contains v, H contains w, both, or v and w are on the same side of the complement of H. Suppose $H = \{x_i = 0\}$ is a coordinate hyperplane for some i. Then $v_i \geq 0$ and $w_i \leq 0$, so they cannot lie in the same component of the complement of H (since these are the halfspaces $x_i > 0$ and $x_i < 0$). This means H must contain at least one of v_i or w_i, or in other words, we must have at least one of $v_i = 0$ or $w_i = 0$. Since this holds for each i, it follows that the non-zero coordinates do not overlap, as claimed.

To see that $u(v) < u(w)$, since the non-zero coordinates do not overlap, i.e., $\mathrm{pos}(v) \cap \mathrm{pos}(w) = \varnothing$, we know that $\#\mathrm{pos}(v) + \#\mathrm{neg}(w) \leq n$. Then $u(v) = \#\mathrm{pos}(v) \leq n - \#\mathrm{neg}(w) < n + 1 - \#\mathrm{neg}(w) = u(w)$.

Define $b = (b_1, \ldots, b_n)$ by

$$b_i = \begin{cases} b_i = v_i & v_i \neq 0 \\ b_i = w_i & w_i \neq 0 \\ b_i = 0 & \text{otherwise} \end{cases}.$$

Since $\mathrm{pos}(v) \cap \mathrm{neg}(w) = \varnothing$, this is well-defined. It is a straightforward linear algebra exercise to see that b lies on the edge of $S_{\mathcal{B}}$ between v and w. □

This gives information on adjacent real vertices, but we can also say something about a real vertex which is adjacent to a fake vertex.

Lemma 5.3. *If b is a fake vertex adjacent to a real vertex a in $S'_{\mathcal{B}}$, then $s(b) > s(a)$.*

Proof. Since b is a fake vertex, it must have both positive and negative coordinates, but since a is real, it can only have one type. If $\mathrm{pos}(a) = \varnothing$, then we cannot have

$\mathrm{pos}(b) \subseteq \mathrm{pos}(a)$, so by the adjacency rules of $S'_{\mathcal{B}}$, we must have $\mathrm{pos}(a) \subsetneq \mathrm{pos}(b)$ and $\mathrm{neg}(a) \subseteq \mathrm{neg}(b)$ (where the first inclusion is proper). Similarly, if $\mathrm{neg}(a) = \varnothing$, then we cannot have $\mathrm{neg}(b) \subseteq \mathrm{neg}(a)$, so we have $\mathrm{pos}(a) \subseteq \mathrm{pos}(b)$ and $\mathrm{neg}(a) \subsetneq \mathrm{neg}(b)$ (where the second inclusion is proper). In either case,

$$s(a) = \#\mathrm{pos}(a) + \#\mathrm{neg}(a) < \#\mathrm{pos}(b) + \#\mathrm{neg}(b) = s(b). \qquad \square$$

For any vertex $b = (b_1, \dots, b_n)$ of $S'_{\mathcal{B}}$, let $b^+ = (b_1^+, \dots, b_n^+)$ be given by $b_i^+ = b_i$ when $b_i \geq 0$ and $b_i^+ = 0$ when $b_i < 0$, and similarly, let $b^- = (b_1^-, \dots, b_n^-)$ be given by $b_i^- = b_i$ when $b_i \leq 0$ and $b_i^- = 0$ when $b_i > 0$. The next Lemma follows immediately from Lemma 5.2.

Lemma 5.4. *If b is a fake vertex, then b^+ and b^- are real vertices which are adjacent in $S_{\mathcal{B}}$, and b lies on the real edge joining b^+ and b^-.*

The following two propositions follows from Theorem 2.14 and Theorem 2.15.

Proposition 5.5. *For vertices $a, b \in S'_{\mathcal{B}}$, say $a \leq_s b$ if a and b are adjacent or equal, and $s(a) \leq s(b)$. Then $((S'_{\mathcal{B}})^0, \leq_s)$ is a poset.*

Proposition 5.6. *For vertices $a, b \in S_{\mathcal{B}}$, say $a \leq_u b$ if a and b are adjacent or equal, and $u(a) \leq u(b)$. Then $((S_{\mathcal{B}})^0, \leq_u)$ is a poset.*

5.2. **The Artin complex.** We can transfer the information of the previous section back to $\Delta'_{\mathcal{B}}$ in order to discuss injectivity. As with the Coxeter complex, vertices inherit an s-type and u-type (Definition 3.9). If v is a vertex of $\Delta_{\mathcal{B}}$, it has a u-type $\hat{u}_i$ and we define $u(v) = i$. If w is a vertex of $\Delta'_{\mathcal{B}}$, it has an s-type $\hat{s}_j$ and we define $s(w) = j$. We will say a cell of $\Delta_{\mathcal{B}}$ is *real*, and say a cell of $\Delta'_{\mathcal{B}}$ which is not a cell of $\Delta_{\mathcal{B}}$ is *fake*. Let $\overline{\cdot} : \Delta'_{\mathcal{B}} \to S'_{\mathcal{B}}$ be the projection map defined previously. For a vertex $v \in \Delta'_{\mathcal{B}}$, we will let $\mathrm{pos}(v) = \mathrm{pos}(\overline{v})$ and $\mathrm{neg}(v) = \mathrm{neg}(\overline{v})$. The following are straightforward exercises.

Proposition 5.7. *Let v be a vertex of $\Delta'_{\mathcal{B}}$.*

(1) *The projection $\Delta'_{\mathcal{B}} \to S'_{\mathcal{B}}$ is injective on real simplices.*
(2) *A cell δ is real (resp., fake) if and only if $\overline{\delta}$ is real (resp., fake).*
(3) *$s(v) = s(\overline{v})$, and if v is real, $u(v) = u(\overline{v})$ (in other words, the projection is type-preserving)*
(4) *v is a real vertex if and only if either* $\mathrm{neg}(v) = \varnothing$ *or* $\mathrm{pos}(v) = \varnothing$. (In the first case, we will call v "non-negative", and in the second, we will call v "non-positive".)
(5) *If v is non-negative, then $u(v) = s(v) = \#\mathrm{pos}(v)$, and if v is non-positive, $u(v) = n + 1 - s(v)$.*

We now need to establish some facts about adjacency in $\Delta_{\mathcal{B}}$ and $\Delta'_{\mathcal{B}}$, with the goal of showing that the relation $\leq$ on $(\Delta'_{\mathcal{B}})^0$ is indeed a partial order.

If v and w are vertices which are adjacent (resp., adjacent or equal) in the unsubdivided complex, we will write $v \sim_u w$ (resp., $v \simeq_u w$). If v and w are adjacent (resp., adjacent or equal) in the subdivided complex, we will write $v \sim_s w$ (resp., $v \simeq_s w$).

Lemma 5.8. *Suppose v and w are vertices of $\Delta'_{\mathcal{B}}$.*

(1) *If v and w are contained in a common real simplex, then $v \simeq_s w$ if and only if either* $\mathrm{pos}(v) \subseteq \mathrm{pos}(w)$ *and* $\mathrm{neg}(v) \subseteq \mathrm{neg}(w)$, *or* $\mathrm{pos}(w) \subseteq \mathrm{pos}(v)$ *and* $\mathrm{neg}(w) \subseteq \mathrm{neg}(v)$.

(2) If both are non-negative (or both non-positive), then v and w are adjacent in $\Delta_{\mathcal{B}}$ if and only if they are adjacent in $\Delta'_{\mathcal{B}}$.
(3) If v is non-negative, w is non-positive, and $v \sim_u w$, then their non-zero coordinates do not overlap. In this case, $v <_u w$, and there is a fake vertex y on the edge between v and w.

Proof. Points (1) and (3) follow immediately from Proposition 5.7(1), Proposition 5.1, and Lemma 5.2. For (2), first suppose v and w are adjacent in $\Delta_{\mathcal{B}}$. Then the real edge e between them injectively maps to the real edge $\overline{e}$ of $S_{\mathcal{B}}$ between the non-negative (or non-positive) vertices $\overline{v}$ and $\overline{w}$ (Proposition 5.7(1), (2), and (4)). Then Lemma 5.2 says there is no fake vertex on $\overline{e}$, so there is no fake vertex on e, and hence $v \sim_s w$.

Conversely, suppose v and w are adjacent in $\Delta'_{\mathcal{B}}$. Let e be the edge of $\Delta'_{\mathcal{B}}$ connecting v and w. Then there is a real simplex $\delta \subseteq \Delta_{\mathcal{B}}$ containing e: if e is a real edge, choose $\delta = e$, and if e is a fake edge, then since $\Delta'_{\mathcal{B}}$ is a (strictly) finer cell structure than $\Delta_{\mathcal{B}}$, it must be properly contained in some real simplex. Then δ maps isomorphically to $\overline{\delta}$ (Proposition 5.7(1)). Then $\overline{v}$ and $\overline{w}$ are both non-negative (or both non-positive) and are adjacent in $S'_{\mathcal{B}}$, so they are adjacent in $S_{\mathcal{B}}$ by Lemma 5.2. In particular, $\overline{e}$ must have been a real edge to start with, since there are no double edges, and thus so is e, implying $v \sim_u w$. □

Lemma 5.9. *If $v \leq_s w$, then $\operatorname{pos}(v) \subseteq \operatorname{pos}(w)$ and $\operatorname{neg}(v) \subseteq \operatorname{neg}(w)$.*

Proof. Suppose to the contrary that either $\operatorname{pos}(v) \not\subseteq \operatorname{pos}(w)$ or $\operatorname{neg}(v) \not\subseteq \operatorname{neg}(w)$. If it is also the case that $\operatorname{pos}(w) \not\subseteq \operatorname{pos}(v)$ or $\operatorname{neg}(w) \not\subseteq \operatorname{neg}(v)$, then $v \not\sim_s w$ (Lemma 5.8), which is a contradiction. So $\operatorname{pos}(w) \subseteq \operatorname{pos}(v)$ and $\operatorname{neg}(w) \subseteq \operatorname{neg}(v)$. Our original contradiction assumption tells us that (at least) one of these inclusions must be proper, so we may assume that $\operatorname{pos}(w) \subsetneq \operatorname{pos}(v)$ (the other inclusion results in an identical conclusion). Then $\#\operatorname{pos}(w) < \#\operatorname{pos}(v)$ and $\#\operatorname{neg}(w) \leq \#\operatorname{neg}(v)$, so

$$s(w) = \#\operatorname{pos}(w) + \#\operatorname{neg}(w) < \#\operatorname{pos}(v) + \#\operatorname{neg}(v) = s(v),$$

which contradicts the fact that $s(v) \leq s(w)$ (from the definition of $\leq_s$). □

Lemma 5.10. *Suppose v is a real vertex and w is a fake vertex. If $v \sim_s w$, then $v <_s w$.*

Proof. The edge between v and w is contained in a real simplex δ. The projection is injective on δ, so $\overline{v}$ and $\overline{w}$ are adjacent in $S'_{\mathcal{B}}$. Moreover, $\overline{v}$ is a real vertex and $\overline{w}$ is a fake vertex. So by Lemma 5.3 and Proposition 5.7(3), $s(v) = s(\overline{v}) < s(\overline{w}) = s(w)$, and thus $v <_s w$. □

Similar to the Coxeter complex, we can define "projections" from fake vertices to distinguished adjacent real vertices.

Lemma 5.11. *Suppose v is a fake vertex. Then there exists a non-negative vertex v^+ and a non-positive vertex v^- such that $v^+ \sim_u v^-$ and v lies on the real edge between them. Moreover, $s(v) = s(v^+) + s(v^-) = u(v^+) - u(v^-) + n + 1$.*

Proof. Let δ be a real simplex containing v. Then δ is combinatorially isomorphic to $\overline{\delta}$ (Proposition 5.7(1)) and $\overline{v}$ is a fake vertex of $S'_{\mathcal{B}}$. Then $\overline{v}$ lies on the real edge $\overline{e}$ between $(\overline{v})^+$ and $(\overline{v})^-$. Since $\overline{\delta}$ is a real simplex and contains an interior point of the real edge $\overline{e}$, it must contain all of e, and in particular, contains $(\overline{v})^+$ and

$(\overline{v})^-$. Letting e, v^+, and v^- be the preimages of $\overline{e}$, $(\overline{v})^+$, and $(\overline{v})^-$, resp., under the isomorphism $\delta \to \overline{\delta}$ and applying Proposition 5.7(3) gives the result. □

Lemma 5.12. *Suppose v is a real vertex, w is a fake vertex, and $v \sim_s w$. If v is non-negative, then $v \leq_u w^+$ and $v \leq_u w^-$. If v is non-positive, then $w^+ \leq_u v$ and $w^- \leq_u v$.*

Proof. We first show that $v \simeq_u w^+$ and $v \simeq_u w^-$. Let e be the real edge between w^+ and w^-, so that w lies in the interior of e. Let e' be the fake edge between v and w. This fake edge must be contained in a real simplex δ. Then this simplex contains the midpoint of the real edge e, so it must contain the whole edge, and in particular, must contain its vertices w^+ and w^-. Since this real simplex also contains v, it follows that $v \simeq_u w^+$ and $v \simeq_u w^-$.

Assume that v is non-negative. Since w^- is non-positive and $v \sim_u w^-$, we know $v \leq_u w^-$ (Lemma 5.8(3)). Since $v \leq_s w$ (Lemma 5.10), then $\text{pos}(v) \subseteq \text{pos}(w) = \text{pos}(w^+)$ (Lemma 5.9). Since v and w^+ are non-negative, $u(v) = \#\text{pos}(v) \leq \#\text{pos}(w^+) = u(w^+)$, so $v \leq_u w^+$.

Now assume that v is non-positive. Since w^+ is non-negative and $v \sim_u w^+$, we know $w^+ \leq_u v$ (Lemma 5.8(3)). Since $v \leq_s w$ (Lemma 5.10), then $\text{neg}(v) \subseteq \text{neg}(w) = \text{neg}(w^-)$ (Lemma 5.9), and in particular, $\#\text{neg}(v) \leq \#\text{neg}(w^-)$. Since v and w^+ are non-positive, $u(v) = n + 1 - \#\text{neg}(v) \geq n + 1 - \#\text{neg}(w^+) = u(w^+)$, so $w^+ \leq_u v$. □

Lemma 5.13. *Suppose v and w are fake vertices. Then $v \leq_s w$ if and only if $v^+ \leq_u w^+ \leq_u w^- \leq_u v^-$.*

Proof. Suppose $v \leq_s w$. Let e_v be the real edge between v^+ and v^-, let e_w be the real edge between w^+ and w^-, and let e_{vw} be the fake edge between v and w. Then e_{vw} must be contained in a real simplex δ. Since δ contains interior points of e_v and e_w, it must contain all of e_v and all of e_w, and in particular, contains v^+, v^-, w^+, and w^-. This means that v^+, v^-, w^+, and w^- are pairwise u-adjacent. Since w^+ is non-negative and v^- is non-positive, $w^+ \leq v^-$. By Lemma 5.9, since $v \leq_s w$,

$$\begin{aligned} \text{pos}(v^+) &= \text{pos}(v) \subseteq \text{pos}(w) = \text{pos}(w^+) = u(w^+), \quad \text{and} \\ \text{neg}(v^-) &= \text{neg}(v) \subseteq \text{neg}(w) = \text{neg}(w^-) = u(w^-). \end{aligned}$$

Then

$$\begin{aligned} u(v^+) &= \#\text{pos}(v^+) \leq \#\text{pos}(w^+) = u(w^+), \quad \text{and} \\ u(v^-) &= n + 1 - \#\text{neg}(v) \geq n + 1 - \text{neg}(w^+) = u(w^-). \end{aligned}$$ □

Conversely, suppose $v^+ \leq_u w^+ \leq_u w^- \leq_u v^-$. Since the u-order is transitive, the vertices v^+, v^-, w^+, and w^- are pairwise u-adjacent. Since $\Delta_{\mathcal{B}}$ is a flag complex [GP12, Prop. 4.5], these vertices span a real simplex. This real simplex contains the real edges which contain v and w, so the simplex contains v and w. Now, since $v^+ \leq_u w^+$ and v^+ and w^+ are non-negative, we have $v^+ \leq_s w^+$ (Lemma 5.8(1) and (2), and Proposition 5.7(5)). Then $\text{pos}(v^+) \subseteq \text{pos}(w^+)$ (Lemma 5.9). Similarly, $w^- \leq_u v^-$, so $w^- \sim_s v^-$ since they are both non-positive (Lemma 5.8(2)), and $u(w^-) \leq u(v^-)$, so $s(v^-) = n + 1 - u(v^-) \leq n + 1 - u(w^-) = s(w^-)$, implying $v^- \leq_s w^-$, so $\text{neg}(v^-) \subseteq \text{neg}(w^-)$. Then

$$\text{pos}(v) = \text{pos}(v^+) \subseteq \text{pos}(w^+) = \text{pos}(w) \qquad \text{and} \qquad \text{neg}(v) = \text{neg}(v^-) \subseteq \text{neg}(w^-) = \text{neg}(w),$$

so since v and w are contained in a real simplex, $v \leq_s w$ by Lemma 5.8(1).

Proposition 5.14. *Let $V = \{v_1, \dots, v_k\}$ be a collection of vertices of $\Delta'_{\mathcal{B}}$ such that $v_i \le v_{i+1}$ for each i. Then there exists a real simplex δ containing V.*

Proof. By Lemma 5.10, there must exist some $0 \le j \le k$ so that v_i is real when $i \le j$ and is fake when $i > j$. Let $V_r = \{v_1, \dots, v_j\}$ denote the (possibly empty) set of real vertices contained in V, and let $V_f = \{v_{j+1} \dots, v_k\}$ denote the (possibly empty) set of fake vertices. Let $V_{f\pm} = \{\, v_i^+, v_i^- : i > j \,\}$ be the real projections of the fake vertices. By Lemma 5.13, for each $j < i < n$, $v_i^+ \le_u v_{i+1}^+ \le_u v_{i+1}^- \le_u v_i^-$. By transitivity of the u-order, this means

$$v_{j+1}^+ \le_u v_{j+2}^+ \le_u \dots \le_u v_k^+ \le v_k^- \le_u v_{k-1}^- \le_u \dots \le_u v_{j+1}^-.$$

The adjacency within the set of real vertices V_r occurs in the subdivided complex $\Delta'_{\mathcal{B}}$, so every vertex of V_r is either non-negative or non-positive. Suppose first that they are non-negative. Then $u(v_i) = s(v_i) < s(v_{i+1}) = u(v_{i+1})$ for every $i < j$, so $v_i \le_u v_{i+1}$. Moreover, since v_j is non-negative and adjacent to the fake vertex v_{j+1}, Lemma 5.12 tells us that $v_j \le_u v_{j+1}^+$. Since the u-order is a partial order, and in particular transitive, this means

$$v_1 \le_u \dots \le_u v_j \le v_{j+1}^+ \le_u \dots \le_u v_k^+ \le v_k^- \le_u v_{k-1}^- \le_u \dots \le_u v_{j+1}^-.$$

In other words, the elements of $V_r \cup V_{f\pm}$ are real vertices which are pairwise connected by real edges. Since the unsubdivided complex is flag, this implies that they span a real simplex. The real edges between the elements of $V_{f\pm}$ contain the fake vertices V_f, so this real simplex contains all of V. The non-positive case is similar. The only difference is that $u(v_i) = n + 1 - s(v_i) > n + 1 - s(v_{i+1}) = u(v_{i+1})$ if $i < j$, so $v_{i+1} \le_u v_i$, but since v_j is non-positive, Lemma 5.12 now tells us that $v_{j+1}^- \le_u v_j$, so we still have a linearly ordered set

$$v_{j+1}^+ \le_u \dots \le_u v_k^+ \le v_k^- \le_u v_{k-1}^- \le_u \dots \le_u v_{j+1}^- \le_u v_j \le_u v_{j-1} \le_u \dots \le_u v_1.$$

Then for the same reasons in the non-negative case, $V_r \cup V_{f\pm}$ spans a real simplex containing V. □

Proposition 5.15. *The relation $\le$ is a partial order on $(\Delta'_{\mathcal{A}(3)})^0 = (\Delta'_{\mathcal{B}})^0$.*

Proof. Clearly $\le$ is reflexive. Showing that $\le$ is antisymmetric amounts to showing that there are no distinct adjacent vertices of the same s-type. Indeed, if $v \sim_s w$ with $v \ne w$, then the edge between v and w is contained in a real simplex δ, on which the projection to $S'_{\mathcal{B}}$ is injective, so $\overline{v}$ and $\overline{w}$ are distinct and adjacent in $S'_{\mathcal{B}}$. Adjacent vertices in the Coxeter complex cannot have the same type, so $s(v) = s(\overline{v}) \ne s(\overline{w}) = s(w)$. This means if $v \le w$ and $w \le v$, then $v \sim_s w$ and $s(v) = s(w)$, thus we must have $v = w$. It remains to show that $\le$ is transitive.

Suppose v, w, z are vertices such that $v \le w$ and $w \le z$. By Proposition 5.14, there is a real simplex δ containing $\{v, w, z\}$. Then $s(v) \le s(w) \le s(z)$, or more specifically, Lemma 5.9 says that $\mathrm{pos}(v) \subseteq \mathrm{pos}(w) \subseteq \mathrm{pos}(z)$ and $\mathrm{neg}(v) \subseteq \mathrm{neg}(w) \subseteq \mathrm{neg}(z)$. Then Lemma 5.8(1) implies $v \sim_s z$, and thus $v \le_s z$, since $s(v) \le s(z)$. □

Proposition 5.16. *$\Delta'_{\mathcal{A}(3)}$ $(= \Delta'_{\mathcal{B}})$ is a flag complex.*

Proof. Suppose V is a collection of vertices which are pairwise adjacent. By Proposition 5.14, there is a real simplex δ containing V. Then $\overline{\delta}$ contains the vertices of $\overline{V}$ and in particular the edges between them. Since the vertices of $\overline{V}$ are pairwise adjacent and $S'_{\mathcal{B}}$ is flag, they span a simplex $\overline{\delta_0}$ of $S'_{\mathcal{B}}$. We know δ is isomorphic to $\overline{\delta}$, so we can pull back $\overline{\delta_0}$ to see that V spans a simplex δ_0 of $\Delta'_{\mathcal{B}}$. □

5.3. **Verifying the link condition.** Now we want to reduce checking the bowtie free and upward flag conditions for the subdivided skewed A_n type arrangements to only needing to verify certain cycles in the 1-skeleton of the subdivided Artin complex can be "filled" in a certain way. We start with Propositions 5.18 and 5.19, which show how to fill these certain cycles, then conclude with Theorem 5.20 to show how filling these cycles implies the full strength of bowtie free and upward flag. First, we will need a technical lemma to make some of the future arguments faster.

Define $(S'_{\mathcal{B}})^+$ to be the full subcomplex of $S'_{\mathcal{B}}$ on the non-negative vertices, and $S^-_{\mathcal{B}}$ to be the full subcomplex on the non-positive vertices. Then let $(\Delta'_{\mathcal{B}})^+$ and $(\Delta'_{\mathcal{B}})^-$ be the pullback of $(S'_{\mathcal{B}})^+$ and $(S'_{\mathcal{B}})^-$ under the projection $\Delta'_{\mathcal{B}} \to S'_{\mathcal{B}}$, respectively. Recall that Proposition 4.4 says that the restriction of the s-order to $(\Delta'_{\mathcal{B}})^+$ is a partial order which is bowtie free and upward flag. The following proposition deals with $(\Delta'_{\mathcal{B}})^-$.

Lemma 5.17. *There is a combinatorial isomorphism $\iota : \Delta'_{\mathcal{B}} \to \Delta'_{\mathcal{B}}$ which is s-order preserving, u-order reversing, and maps $(\Delta'_{\mathcal{B}})^+$ isomorphically to $(\Delta'_{\mathcal{B}})^-$ (and vice versa). In particular, the restriction of the s-order to $(\Delta'_{\mathcal{B}})^-$ is a partial order which is bowtie free and upward flag.*

Proof. Consider $-(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, the usual inversion map $p \mapsto -p$ on $\mathbb{R}^n$. This restricts to a combinatorial automorphism of $S_{\mathcal{B}}$ which preserves the subdivision $S'_{\mathcal{B}}$, and a combinatorial isomorpism of $\Sigma_{\mathcal{B}}$. We can extend this to an automorphism of the Salvetti complex $\widehat{\Sigma}_{\mathcal{B}}$ as follows. The map on the vertices is still given simply by the inversion map (as the vertices of $\widehat{\Sigma}_{\mathcal{B}}$ are the vertices of $\Sigma_{\mathcal{B}}$). Suppose F is a face of $\Sigma_{\mathcal{B}}$ and v is a vertex of F, so $[F, v]$ is a face of $\widehat{\Sigma}_{\mathcal{B}}$. Then we define $-[F, v] = [-F, -v]$. It is easy to see that this map gives a well-defined automorphism of $\widehat{\Sigma}_{\mathcal{B}}$. Let $\rho : \widetilde{K} \to \widehat{\Sigma}_{\mathcal{B}}$ be the universal cover of $\widehat{\Sigma}_{\mathcal{B}}$, and let $\kappa : \widetilde{K} \to \widetilde{K}$ be the lift of the inversion map $-(\cdot) : \widehat{\Sigma}_{\mathcal{B}} \to \widehat{\Sigma}_{\mathcal{B}}$ to $\widetilde{K}$. In particular, $\rho \circ \kappa = -\rho$.

We now define $\iota : \Delta_{\mathcal{B}} \to \Delta_{\mathcal{B}}$ on the unsubdivided complex. Let $\mathcal{F}$ be the collection of faces of $S_{\mathcal{B}}$, and for $F \in \mathcal{F}$, let Λ_F be an index set of the elevations of the standard subcomplex $\widehat{F}$ (cf. Definition 3.1 and Section 3.2). First we define ι_0 as an automorphism of $\bigsqcup_{F \in \mathcal{F}} F \times \Lambda_F$. For every $F \in \mathcal{F}$, there is a bijection $\Lambda_F \to \Lambda_{-F}$, which we will denote $\lambda \mapsto -\lambda$, induced by κ. More specifically, since κ commutes with the projection, it bijectively maps the elevations of F to the elevations of $-F$ in such a way that preserves nesting of elevations, and this induces the map on the index sets. Then we define $\iota_0(F \times \{\lambda\}) = (-F) \times \{-\lambda\}$. By our definition of the bijection $\Lambda_F \to \Lambda_{-F}$, it is clear that this respects the gluing of the Falk complex, and thus descends to an automorphism ι on $\left(\bigsqcup_{F \in \mathcal{F}} F \times \Lambda_F\right) / \sim = \Delta_{\mathcal{B}}$. Letting $\overline{\cdot} : \Delta_{\mathcal{B}} \to S_{\mathcal{B}}$ be the usual projection, it is immediate from the definitions that $\overline{\iota(x)} = -\overline{x}$ for all $x \in \Delta_{\mathcal{B}}$. In particular, ι preserves the subdivision $\Delta'_{\mathcal{B}}$.

Thus we see for a vertex $v \in \Delta'_{\mathcal{B}}$ that $s(v) = s(\overline{v}) = s(-\overline{v}) = s(\overline{\iota(v)}) = s(\iota(v))$, but for a vertex $v \in \Delta_{\mathcal{B}}$ that $u(v) = u(\overline{v}) = n + 1 - u(-\overline{v}) = n + 1 - u(\overline{\iota(v)}) = n + 1 - u(\iota(v))$. So if $v, w \in (\Delta'_{\mathcal{B}})^0$ and $v \leq_s w$, then $\iota(v) \sim_s \iota(w)$ since ι is an automorphism, and $s(\iota(v)) = s(v) \leq s(w) = s(\iota(w))$, so $\iota(v) \leq_s \iota(w)$. So ι is s-order preserving. But if $v, w \in (\Delta_{\mathcal{B}})^0$ and $v \leq_u w$, then $\iota(v) \sim_u \iota(w)$ since ι is an automorphism, while $u(\iota(v)) = n + 1 - u(v) \geq n + 1 - u(w) = u(\iota(w))$, so $w \leq_u v$.

So ι is u-order reversing. In particular, it is an order-preserving isomorphism of $(((\Delta'_{\mathcal{B}})^+)^0, <_s)$ and $(((\Delta'_{\mathcal{B}})^-)^0, <_s)$, so $(\Delta'_{\mathcal{B}})^-$ inherits all the s-order properties enjoyed by $(\Delta'_{\mathcal{B}})^+$. □

Proposition 5.18. *Any embedded* 4*-cycle in* $\Delta'_{\mathcal{B}}$ *of* s*-type* $1n1n$ *has a central vertex, i.e., a vertex which is* s*-adjacent or equal to each vertex of the cycle.*

Proof. Let (v_1, w_1, v_2, w_2) be an embedded 4-cycle in $\Delta'_{\mathcal{B}}$ with $s(v_i) = 1$ and $s(w_i) = n$ for $i = 1, 2$. For $i = 1, 2$, since $\#\mathrm{pos}(v_i) + \#\mathrm{neg}(v_i) = s(v_i) = 1$, this means either $\mathrm{pos}(v_i) = \varnothing$ or $\mathrm{neg}(v_i) = \varnothing$. In particular, the v_i must be real vertices.

First, assume $\mathrm{neg}(v_1) = \mathrm{neg}(v_2) = \varnothing$. Choose some $i = 1, 2$. If w_i is real, then since it's s-adjacent to the non-negative vertices v_1 and v_2, it is also non-negative (Lemma 5.8(3)), and in particular, is u-adjacent to these vertices (Lemma 5.8(2)). If w_i is not real, then w_i^+ is non-negative (by definition) and $v_j \leq_u w_i^+$ by Lemma 5.12. We can summarize this as follows: for $i = 1, 2$, if w_i is not fake, define $w'_i = w_i$, and if w_i is fake, define $w'_i = w_i^+$. Then w'_i is non-negative and $v_i \leq_u w'_j$ for $i, j = 1, 2$. Then by Proposition 4.4, there is a vertex $v \in (\Delta'_{\mathcal{B}})^+$ with $v_i \leq_u v$ and $v \leq_u w'_i$. In particular, $v_i \leq_s v$ and $v \leq_s w'_i$ by Proposition 5.7(5). If w_i is real then $v \leq_s w'_i = w_i$. If it is fake, then $w'_i = w_i^+ \leq_s w_i$ by Lemma 5.10. Since the s-order is transitive, $v \leq_s w_i$.

Next, assume $\mathrm{pos}(v_1) = \mathrm{pos}(v_2) = \varnothing$. This case follows immediately from Lemma 5.17, but we provide the details for completeness. Choose some $i = 1, 2$. If w_i is real, then since it's s-adjacent to the non-positive vertices v_1 and v_2, it is also non-positive (Lemma 5.8(3)). Note we have already assumed that $v_j \leq_s w_i$, so $\iota(v_j) \leq_s \iota(w_i)$. If w_i is not real, then w_i^- is non-positive (by definition) and $w_i^- \leq_u v_j$ by Lemma 5.12. But then since these vertices live in $(\Delta'_{\mathcal{B}})^-$, we know from Lemma 5.17 that $\iota(v_j) \leq_u \iota(w_i^-)$ since ι is u-order reversing, and consequently $\iota(v_j) \leq_s \iota(w_i^-)$ since their images under ι are non-negative, and the u- and s-orders agree on the non-negative part. We can summarize this as follows: for $i = 1, 2$, if w_i is not fake, define $w'_i = w_i$, and if w_i is fake, define $w'_i = w_i^+$. Then w'_i is non-negative and $\iota(v_i) \leq_s \iota(w'_j)$ for $i, j = 1, 2$. Then by Proposition 4.4, there is a vertex $v \in (\Delta'_{\mathcal{B}})^+$ with $\iota(v_i) \leq_s v$ and $v \leq_s \iota(w'_i)$. In particular $v_i \leq_s \iota^{-1}(v)$ and $\iota^{-1}(v) \leq_s w'_i$ since ι is s-order preserving. If w_i is real then $\iota^{-1}(v) \leq_s w'_i = w_i$. If it is fake, then $w'_i = w_i^- \leq_s w_i$ by Lemma 5.10. Since the s-order is transitive, $\iota^{-1}(v) \leq_s w_i$.

Now assume $\mathrm{neg}(v_1) = \mathrm{pos}(v_2) = \varnothing$ (the case $\mathrm{pos}(v_1) = \mathrm{neg}(v_2) = \varnothing$ is identical). Note that for $i = 1, 2$, w_i cannot be a real vertex: if it was non-negative, then it could not be s-adjacent to the non-positive vertex v_2, and if it were non-positive, it could not be s-adjacent to the non-negative vertex v_1 (Lemma 5.8(3)). So both w_1 and w_2 are fake vertices.

We claim that $v_1 \leq_u v_2$. Indeed, since v_1 and v_2 are s-adjacent to the fake vertex w_1, then $v_1 \leq_u w_1^+$ since v_1 is non-negative, and $w_1^+ \leq_u v_2$ since v_2 is non-positive (Lemma 5.12). By transitivity of the u-order, $v_1 \leq_u v_2$, as claimed. Then by Lemma 5.8(3), there is a (fake) vertex v (possibly equal to either w_1 or w_2) which lies on the real edge between v_1 and v_2. Then Lemma 5.11 implies that $v^+ = v_1$ and $v^- = v_2$ (the real edge that v lies on is clearly unique); in particular, $v \sim_s v_i$ for $i = 1, 2$. We have already seen that $v^+ = v_1 \leq_u w_1^+$. For identical reasons (namely, Lemma 5.12), we also have $v^+ = v_1 \leq_u w_2^+$, but also $w_1^- \leq_u v_2 = v^-$ and $w_2^- \leq_u v_2 = v^-$. Note that it is always the case that $w_1^+ \leq_u w_1^-$ and $w_2^+ \leq_u w_2^-$

(Lemma 5.8(3)). So in summary, for $i = 1, 2$, $v^+ \leq_u w_i^+ \leq_u w_i^- \leq_u v^-$. Thus by Lemma 5.13, $v \sim_s w_i$ for $i = 1, 2$. □

Proposition 5.19. *Any embedded* 6*-cycle in* $\Delta'_{\mathcal{B}}$ *of s-type* $1n1n1n$ *has a vertex in* $\Delta'_{\mathcal{B}}$ *s-adjacent to all the s-type* 1 *vertices.*

Proof. Let $(v_1, w_1, v_2, w_2, v_3, w_3)$ be an embedded 6-cycle in $\Delta'_{\mathcal{B}}$ with $s(v_i) = 1$ and $s(w_i) = n$ for $i = 1, 2, 3$. For each i, Since $s(v_i) = 1$, then either $\mathrm{pos}(v_i) = \varnothing$ or $\mathrm{neg}(v_i) = \varnothing$, so the v_i must be real vertices. We will take cases on which v_i are non-positive and which are non-negative.

Suppose first that all v_i are non-negative. This means $1 = s(v_i) = u(v_i) = \#\mathrm{pos}(v_i)$ for each i. For $i = 1, 2, 3$, if w_i is real, define $w'_i = w_i$, and if w_i is fake, define $w'_i = w_i^+$. By Lemma 5.12, if w_i is fake and $v_j \leq_s w_i$ then $v_j \leq_u w_i^+$, and since v_j and w_i^+ are non-negative, $v_j \leq_s w_i^+$. So regardless if w_i is fake or not, we always have $v_j \leq_s w'_i$ whenever $v_j \leq_s w_i$. This means the non-negative v_j are pairwise s-upper bounded by the non-negative w'_i, and Proposition 4.4 tells us they have a common upper bound in the subcomplex $(\Delta'_{\mathcal{B}})^+$, hence in $\Delta'_{\mathcal{B}}$.

Suppose exactly one v_i is non-positive and the rest are non-negative. We may assume that v_3 is non-positive. By a similar argument to the one given in the proof of Proposition 5.18, w_2 and w_3 are fake vertices. We define w'_1 similarly to before: if w_1 is real, define $w'_1 = w_1$, and if it's fake, define $w'_1 = w_1^+$. Notice that $1 = s(v_i) = u(v_i) = \#\mathrm{pos}(v_i)$ for $i = 1, 2$, and $u(v_3) = n + 1 - s(v_3) = n$. Then v_3 is u-adjacent to both v_1 and v_2 in $\Delta_{\mathcal{B}}$: since v_2 is non-negative, we have $v_2 \leq_u w_2^+$, and since v_3 is non-positive, we have $w_2^+ \leq_u v_3$ (Lemma 5.12). By transitivity of the u-order, $v_2 \leq_u v_3$. The argument for v_1 is identical, replacing w_2^+ with w_3^+. Now the 4-cycle (v_1, w'_1, v_2, v_3) in $\Delta_{\mathcal{B}}$ is a bowtie, so there is a central (real) vertex v which is u-adjacent to each vertex, or more specfically, $v_1 \leq_u v$, $v_2 \leq_u v$, $v \leq_u w'_1$, and $v \leq_u v_3$. In particular, $1 < u(v) < n$. There are two possibilities.

Suppose v is non-negative. Then v is adjacent to v_1 and v_2 in $\Delta'_{\mathcal{B}}$ and since these vertices are non-negative, $v_1 \leq_s v$ and $v_2 \leq_s v$. But there is a fake vertex v' on the edge between v and v_3. (In particular, $v' \sim_s v_3$.) Then $v \leq_s v'$ by Lemma 5.10. By transitivity of the s-order, $v_1 \leq_s v'$ and $v_2 \leq_s v'$, so v' is adjacent to each of v_1, v_2, and v_3.

Suppose v is non-positive. Since it is u-adjacent to the non-negative vertex w'_1, there is a fake vertex v' on the real edge between v and w'_1 (Lemma 5.8(3)). In particular, $w'_1 \leq_s v'$ and $v \leq_s v'$. For $i = 1, 2$, since $v_i \leq_s w'_1$, we have $v_i \leq v'$. Last, $v \leq_u v_3$, but since these vertices are non-positive, $v_3 \leq_s v$. Then by transitivity, $v_3 \leq_s v'$.

The cases with exactly two v_i non-positive and with all v_i non-positive are identical to these cases after applying Lemma 5.17. □

Theorem 5.20. $((\Delta'_{\mathcal{A}(3)})^0, <)$ *is bowtie-free and upward flag.*

Proof. In order to make the dimension clear, we let $\mathcal{B}_k$ denote the arrangement $\mathcal{A}(3)$ in $\mathbb{R}^k$. We start by showing $((\Delta'_{\mathcal{B}_k})^0, <)$ is bowtie-free by induction on k. When $k = 1$, there is nothing to show. So assume $k \geq 2$ and $((\Delta'_{\mathcal{B}_j})^0, <)$ is bowtie-free for all $j < k$.

Let $\{v_1, v_2, v_3, v_4\}$ be a set of vertices in $\Delta'_{\mathcal{B}_k}$ with $v_1 <_s v_2$, $v_2 >_s v_3$, $v_3 <_s v_4$, and $v_4 >_s v_1$. Let $\gamma = (v_1, v_2, v_3, v_4)$. If γ is not embedded, then the vertices are not pairwise distinct, so γ has a central vertex (i.e., a vertex which is s-adjacent or equal to each vertex of the cycle). So suppose γ is embedded. For $i = 1, 3$, choose a vertex $v_i' \simeq_s v_i$ with $s(v_i') = 1$. Similarly, for $i = 1, 3$, choose a vertex $v_i' \simeq_s v_i$ with $s(v_i') = k$. Since γ is embedded, we can choose such vertices so that $v_1' \neq v_3'$ and $v_2' \neq v_4'$. By transitivity of the s-order, $v_1' <_s v_2'$, $v_2' >_s v_3'$, $v_3' <_s v_4'$, and $v_4' >_s v_1'$. In particular, $\gamma' = (v_1', v_2', v_3', v_4')$ is an embedded 4-cycle of s-type $1k1k$. Proposition 5.18 guarantees the existence of a central vertex w' for γ'. We will now show how w' gives rise to a central vertex of γ, showing that $\{v_1, v_2, v_3, v_4\}$ is not a bowtie.

If $v_1' = v_1$, define $w_1 = w'$. If not, then note that $v_1 <_s v_i \leq_s v_i'$ for $i = 2, 4$. It follows from transitivity that $v_1 <_s v_i'$ for $i = 2, 4$. So, in $\mathrm{lk}(v_1', \Delta'_{\mathcal{B}_k})$, there is a 4-cycle $\gamma_1 = (v_1, v_2', w', v_4')$. Since $s(v_1') = 1$, v_1' is a real vertex, and $\mathrm{lk}(v_1', \Delta'_{\mathcal{B}_k}) \cong \Delta'_{\mathcal{B}_{k-1}}$. By induction, this complex is bowtie free, so γ_1 has a central vertex in $\mathrm{lk}(v_1', \Delta'_{\mathcal{B}_k})$, which we will call w_1.

If $v_2' = v_2$, define $w_2 = w_1$. If not, then, for similar reasons as before, $v_3' <_s v_2$. So, in $\mathrm{lk}(v_2', \Delta'_{\mathcal{B}_k})$, there is a 4-cycle $\gamma_2 = (v_1, v_2, v_3', w_1)$. Since $s(v_2') = k$, it is either a real vertex or lies on the midpoint of the real line between $(v_2')^+$ and $(v_2')^-$. By Lemma 2.12, we can summarize this by saying $\mathrm{lk}(v_2', \Delta'_{\mathcal{B}_k})$ is isomorphic to the s-subdivision of the join $\Delta_{\mathcal{B}_i} \circ \Delta_{\mathcal{B}_j} \circ \Delta_{\mathcal{B}_\ell}$ where each $0 \leq i, j, \ell < k$ with at least one of i, j, and k positive, and if any are 0, we exclude the corresponding complex from the join. By induction, each of these join factors are bowtie free, so it follows from Lemma 3.11 and its proof that their s-subdivided join $\mathrm{lk}(v_2', \Delta'_{\mathcal{B}_k})$ is bowtie free. So, γ_2 has a central vertex in $\mathrm{lk}(v_2', \Delta'_{\mathcal{B}_k})$, which we will call w_2.

If $v_3' = v_3$, define $w_3 = w_2$. If not, then in $\mathrm{lk}(v_3', \Delta'_{\mathcal{B}_k})$, there is a 4-cycle (w_2, v_2, v_3, v_4'). By an argument similar to the one given for γ_1, this cycle has a central vertex in $\mathrm{lk}(v_3', \Delta'_{\mathcal{B}_k})$, which we will call w_3. Note that $w_2 <_s v_2$, so we must have $w_2 <_s w_3$. In particular, since $v_1 <_s w_2$, we also have $v_1 <_s w_3$.

If $v_4' = v_4$, define $w = w_3$. If not, then in $\mathrm{lk}(v_4', \Delta'_{\mathcal{B}_k})$, there is a 4-cycle (v_1, w_3, v_3, v_4). By an argument similar to the one given for γ_2, this cycle has a central vertex in $\mathrm{lk}(v_4', \Delta'_{\mathcal{B}_k})$, which we will call w. Note that $v_1 <_s w$, so we must have $w <_s w_3$. In particular, since $w_3 <_s v_2$, we also have $w <_s v_2$. Thus w is a central vertex for γ, so $\Delta'_{\mathcal{B}_k}$ is bowtie-free.

Next we show $((\Delta'_{\mathcal{B}_k})^0, <)$ is upward flag by induction on k. If $k = 1$ then there is nothing to show. Assume $k \geq 2$ and $((\Delta'_{\mathcal{B}_j})^0, <_s)$ is upward flag for all $j < k$. Suppose $\{w_1, w_2, w_3\}$ is a collection of pairwise s-upper bounded vertices of $\Delta'_{\mathcal{B}_k}$, with z_i an upper bound for $\{w_i, w_{i+1}\}$ for $i = 1, 2, 3$ (with indices taken cyclically). For each $i = 1, 2, 3$, we may assume that $s(z_i) = k$ (for example, by replacing z_i with any vertex v of s-type k with $z_i \leq_s v$). We will repeat a procedure similar to the one used for 4-cycles to determine an upper bound of the w_i.

For each $i = 1, 2, 3$, let w_i' be a vertex with $w_i' \leq_s w_i$ and $s(w_i') = 1$. (Note that if $s(w_i) = 1$, this implies $w_i' = w_i$.) Then $\{w_1', w_2', w_3'\}$ is a collection of s-type 1 vertices pairwise upper bounded by s-type k vertices, giving a 6-cycle $\beta = (w_1', z_1, w_2', z_2, w_3', z_3)$ of s-type $1k1k1k$. If β is not embedded, then either the w_i' are not pairwise distinct or the z_i are not pairwise distinct. If the w_i' cannot be chosen to be pairwise distinct, then the w_i could not have been pairwise distinct to start with, so one of the z_i will be an upper bound of each vertex and we are done. If the z_i are not pairwise distinct, then one of them is an upper bound for each of the w_i and we are done. So we may assume that the 6-cycle β is embedded. By

Proposition 5.19, there exists a vertex v_0 which is an upper bound of the w_i'. We now find a vertex v which is an upper bound of the w_i.

If $s(w_1) = 1$ (so $w_1' = w_1$), then let $v_1 = v_0$. Otherwise, define v_1 as follows. For each i, consider the 4-cycle $\gamma_i^{(1)} = (w_i', v_0, w_{i+1}', z_i)$. Since $((\Delta'_{\mathcal{B}_k})^0, <_s)$ is bowtie-free by our above work, there must be some vertex v_i' such that $w_i' \le_s v_i'$, $w_{i+1}' \le_s v_i'$, $v_i' \le_s v_0$, and $v_i' \le_s z_i$. Now inside $\mathrm{lk}(w_1', \Delta'_{\mathcal{B}_k})$, the vertices $\{w_1, v_1', v_3'\}$ are pairwise s-upper bounded: z_{i-1} is an upper bound for $\{v_3', w_1\}$, z_1 is an upper bound for $\{w_1, v_1\}$, and v_0 is an upper bound for $\{v_1', v_3'\}$. Similar to the bowtie free case previously, this link is the s-subdivision of the join of posets, each of which are upward flag by induction, and thus is itself upward flag. So, there is an s-upper bound v_1 for $\{w_1, v_1', v_3'\}$ in the link of w_1'. In particular, $w_2' \le_s v_1' \le_s v_1$ and $w_3' \le_s v_3' \le_s v_1$, so v_1 is an upper bound of $\{w_1, w_2', w_3'\}$.

We now repeat a similar procedure on the vertices $\{w_1, w_2', w_3'\}$. If $s(w_2) = 1$, define $v_2 = v_1$; otherwise, define v_2 as follows. Let $\gamma_1^{(2)} = (w_1, v_0, w_2', z_1)$ and $\gamma_i^{(2)} = (w_i', v_0, w_{i+1}', z_i)$ for $i = 2, 3$. To simplify notation, let v_1' now denote a vertex such that $w_1 \le_s v_1'$, $w_2' \le_s v_1'$, $v_1' \le_s v_0$, and $v_1' \le_s z_1$, and for $i = 2, 3$, let v_i' now denote a vertex such that $w_i' \le v_i'$, $w_{i+1}' \le v_i'$, $v_i' \le v_0$, and $v_i' \le z_i$ (these vertices exist because $\Delta'_{\mathcal{B}_k}$ is bowtie-free under the s-order). Now in $\mathrm{lk}(w_2'\Delta'_{\mathcal{B}_k})$, $\{w_2, v_1', v_2'\}$ is pairwise s-upper bounded: u_1 is an upper bound of $\{w_2, v_1'\}$, v_1 is an upper bound of $\{v_1', v_2'\}$, and u_2 is an upper bound of $\{v_2', w_2\}$. So $\{w_2, v_1', v_2'\}$ has an s-upper bound v_2. Note that $w_1 \le_s v_1' \le_s v_2$ and $w_3' \le_s v_2' \le_s v_2$, so v_2 is an upper bound of $\{w_1, w_2, w_3'\}$.

We conclude by examining the vertices $\{w_1, w_2, w_3'\}$. If $s(w_3) = 1$, define $v = v_2$. In this case, $v = v_2$ is already an upper bound of $\{w_1, w_2, w_3\}$. Otherwise, define v as follows. Let $\gamma_1^{(3)} = (w_1, v_0, w_2, z_1)$, $\gamma_2^{(3)} = (w_2, v_0, w_3', z_2)$, and $\gamma_3^{(3)} = (w_3', v_0, w_1, z_3)$. To simplify notation again, let v_1' now denote a vertex such that $w_1 \le_s v_1'$, $w_2 \le_s v_1'$, $v_1' \le_s v_0$, and $v_1' \le_s u_1$, let v_2' now denote a vertex such that $w_2 \le_s v_2'$, $w_3' \le_s v_2'$, $v_2' \le_s v_2$, and $v_2' \le_s u_2$, and let v_3' now denote a vertex such that $w_3' \le_s v_3'$, $w_1 \le_s v_3'$, $v_3' \le_s v_0$, and $v_3' \le_s u_3$. (These all exist since $\Delta'_{\mathcal{B}}$ is bowtie free.) Now in $\mathrm{lk}(w_3'\Delta'_{\mathcal{B}_k})$, u_2 is an (s-)upper bound of $\{w_3, v_2'\}$, v_2 is an upper bound of $\{v_2', v_3'\}$, and u_3 is an upper bound of $\{v_3', w_3\}$. So $\{w_3, v_2', v_3'\}$ has an s-upper bound v. Note that $w_1 \le_s v_3' \le_s v_3$ and $w_2 \le_s v_2' \le_s v_3$, so v is an s-upper bound of $\{w_1, w_2, w_3\}$. □

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(K. Goldman) Department of Mathematics and Statistics, McGill University, Burnside Hall, 805 Sherbrooke Street West, Montreal, QC, H3A 0B9, Canada

Email address: kat.goldman@mcgill.ca

(J. Huang) Department of Mathematics, The Ohio State University, 231 W. 18th Ave, Columbus, OH 43210, U.S.

Email address: huang.929@osu.edu